ON FOX COLORINGS OF SYMMETRIC UNIONS

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Abstract. A symmetric union in the 3-space $\mathbb{R}^3$ is a knot, obtained from a knot in $\mathbb{R}^3$ and its mirror image, which are symmetric with respect to a 2-plane in $\mathbb{R}^3$, by taking the connected sum of them and moreover by connecting them with some vertical twists along the plane. In this paper, we study the p-coloring of a symmetric union.

Key words: Symmetric union; p-coloring; partial knot.

1. Introduction

A *symmetric union* was introduced by Kinoshita and Terasaka [2]. A symmetric union is known to be a *ribbon knot* which bounds a smooth disk in the 4-ball. (See [5] for the definition.) Every ribbon knot with minimal crossing number ≤ 10 is a symmetric union [1, 3] and it is known that all two-bridge ribbon knots can be represented as symmetric unions. (See [4, 6].)

A *p-coloring* of a diagram D of a knot is an assignment of one of the numbers $0, 1, \ldots, p-1$ to each arc of D such a way that at each crossing the sum of the numbers of the under-crossings is equal to twice the number of the over-crossing modulo p. The number of p-colorings of D is a knot invariant. We denote it by $c_p(K)$ for a knot K. A p-coloring is said to be *non-trivial* if it has at least two numbers. A knot K is *p-colorable* if K has a diagram which admits a non-trivial p-coloring.

In this paper, we study p-colorings of a symmetric union and show the following theorem.

Theorem 1.1. *Let $\hat{K}$ be a knot with a symmetric union presentation with a partial knot K and p, a positive odd prime integer. Then K is p-colorable if and only if $\hat{K}$ is p-colorable.*

Corollary 1.2. *Let $\hat{K}$ be a knot with a symmetric union presentation with a partial knot K and p, a positive odd prime integer. Then $c_p(K) = p$ if and only if $c_p(\hat{K}) = p$.*

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2. The definition of a symmetric union

Let $\mathbb{R}^3$ be the 3-space with x-, y-, and z-axes. Let $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}$ and $\mathbb{R}^3_- = \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\}$. Throughout this paper, a *tangle* denotes two arcs properly embedded in a 3-ball. We denote the tangle made of $|m|$ half-twists along the z-axis as a diagram by an integer $m \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ with respect to the x-axis as in Figure 1. A symmetric union is defined as follows.

Definition 2.1. We take a knot $\tilde{K}$ in $\mathbb{R}^3_-$ and its mirror image $\tilde{K}^*$ in $\mathbb{R}^3_+$ such that $\tilde{K}$ and $\tilde{K}^*$ are symmetric with respect to the yz-plane $\mathbb{R}^2_{yz}$ as in Figure $2(a)$. Here we consider a diagram of a knot in the xz-plane $\mathbb{R}^2_{xz}$ and we denote the diagrams of $\tilde{K}$ and $\tilde{K}^*$ by $\tilde{D}$ and $\tilde{D}^*$, respectively. We take 0-tangles $T_0, T_1, \ldots, T_k$ as in Figure $2(a)$. Then we replace the tangles $T_0, T_1, \ldots, T_k$ with tangles ∞, $m_1, m_2, \ldots, m_k$ as in Figure $2(b)$. (See Figure 3 for example.) Here we assume that $m_i \neq \infty$ $(1 \leq i \leq k)$. The resultant diagram is called a *symmetric union presentation* and we denote it by $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$. The knot $\tilde{K}$ is called the *partial knot* of the symmetric union.

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If a knot K has a diagram $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$, then the knot K is called a *symmetric union*.

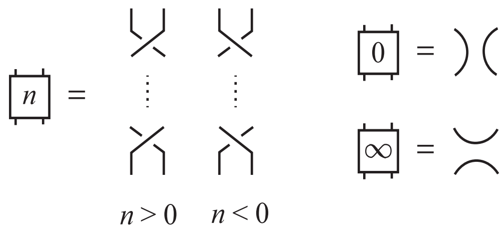

FIGURE 1

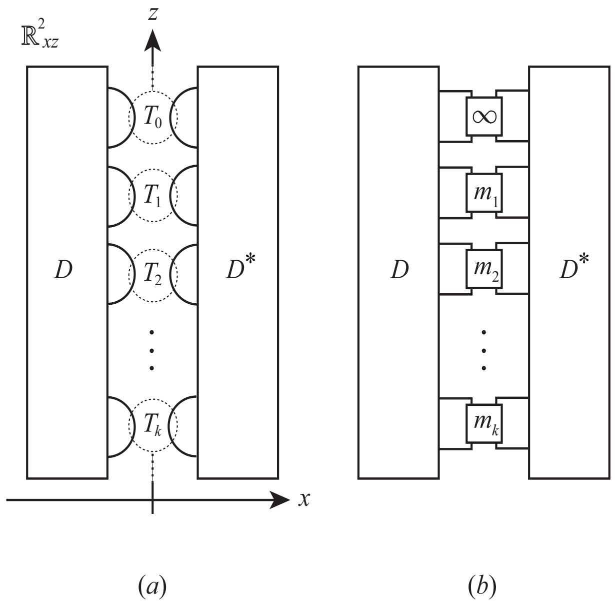

FIGURE 2

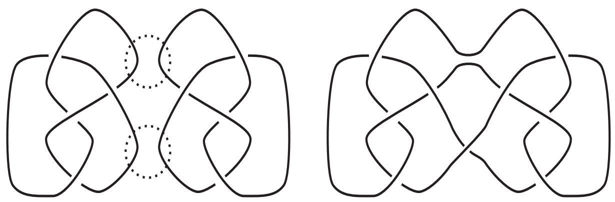

FIGURE 3

3. PROOFS

Proof of Theorem 1.1. By a result of [3], we know that $\det(\hat{K}) = (\det(K))^2$. Note that $\det(\hat{K})$ is divisible by p if and only if $\det(K)$ is divisible by p. On the other hand, it is known that a knot is p-colorable if and only if p devides its determinant when p is an odd prime integer. [7] Thus we know that $\hat{K}$ is p-colorable if and only if K is p-colorable.

4. The number of colorings of a knot

We can easily show the following result since we can obtain a p-coloring of a symmetric union from a p-coloring of the partial knot.

Proposition 4.1. *Let $\hat{K}$ be a knot with a symmetric union presentation with a partial knot K and p, a positive integer. Then $c_p(K) \leq c_p(\hat{K})$.*

Proposition 4.2. [8] *We have $c_p(K_1)c_p(K_2) = p \cdot c_p(\hat{K})$ if $\hat{K}$ is the connected sum of K_1 and K_2.*

Now we consider the following question.

Question. Let $\hat{K}$ be a knot with a symmetric union presentation with a partial knot K. Does the equality $c_p(K)^2 = p \cdot c_p(\hat{K})$ always hold?

Example 4.3. We take a p-coloring of 6_1 as in Figure 5.

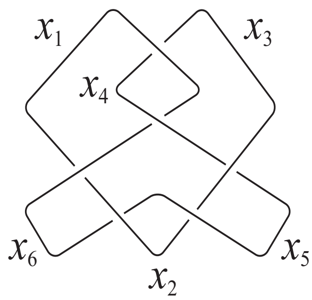

Figure 4

Then we have the following congruences:
$2x_1 \equiv x_3 + x_4 \pmod{p}$
$2x_4 \equiv x_6 + x_1 \pmod{p}$
$2x_6 \equiv x_1 + x_2 \pmod{p}$
$2x_2 \equiv x_6 + x_5 \pmod{p}$
$2x_5 \equiv x_2 + x_3 \pmod{p}$
$2x_3 \equiv x_4 + x_5 \pmod{p}$
Thus we have the following congruences:
$-x_3 + 6x_5 - 5x_6 \equiv 0 \pmod{p}$
$-x_1 + 4x_5 - 3x_6 \equiv 0 \pmod{p}$
$-x_2 - 4x_5 + 5x_6 \equiv 0 \pmod{p}$
$-9x_5 + 9x_6 \equiv 0 \pmod{p}$
$-x_4 + 2x_5 - x_6 \equiv 0 \pmod{p}$
$9x_5 - 9x_6 \equiv 0 \pmod{p}$

In the case when $p \neq 3m$, we have $x_5 \equiv x_6$ by the fourth congruence. Thus we have $x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv x_5 \equiv x_6$ by the other congruences. Therefore we know that $c_p(6_1) = p$.
In the case when $p = 3m$ and $m \neq 3n$, we have $x_5 \equiv x_6 \pmod{m}$ by the fourth congruence. Then we have $9m$ choices for the pair (x_5, x_6). Thus we know that $c_p(6_1) = 9m = 3p$.
In the case when $p = 3m$ and $m = 3n$, we have $x_5 \equiv x_6 \pmod{n}$ by the fourth congruence. Then we have $81n$ choices for the pair (x_5, x_6). Thus we know that $c_p(6_1) = 81n = 9p$.

Thus we have

(1) $$c_p(6_1) = \begin{cases} 9p & (p = 9n\ (n \in \mathbb{N})) \\ 3p & (p = 3m, m \neq 3n\ (m, n \in \mathbb{N})) \\ p & (p \neq 3m\ (m \in \mathbb{N})) \end{cases} .$$

On the other hand, 6_1 has a symmetric union presentation with 3_1 as the partial knot as follows:

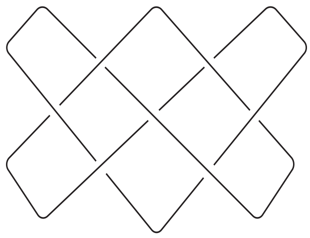

Figure 5

We can easily calculate $c_p(3_1)$ as follows:

(2) $$c_p(3_1) = \begin{cases} 3p & (p = 3m\ (m \in \mathbb{N})) \\ p & (p \neq 3m\ (m \in \mathbb{N})) \end{cases} .$$

So we know that $c_p(3_1)^2 \neq p \cdot c_p(6_1)$ if $p = 3m, m \neq 3n\ (m, n \in \mathbb{N})$. By using Proposition 4.2, we can obtain infinitely many symmetric unions with such a property.

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