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A formula for the Jones polynomial of symmetric unions

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Abstract. Symmetric unions are links obtained from a knot in the 3-space $\mathbb{R}^3$ by taking diagrams of the knot and its mirror image, which are symmetrically placed with respect to an axis in the 2-plane, and connecting them to obtain a diagram by inserting a 2-tangle with no twists and a finite number of 2-tangles with twists along the axis in the diagram. The Jones polynomial of a symmetric union has some interesting properties. In this paper, we give a formula of the Jones polynomial of a symmetric union.

1 Introduction

A *symmetric union* was first introduced by Kinoshita and Terasaka [4]. In recent years, Lamm [5] generalized the definition and investigated the relationship between a symmetric union and a ribbon knot. It is easy to see that every symmetric union represents a ribbon knot, on the other hand, the converse question is still open. In this paper, we give the following formula for the Jones polynomial [2] of a knot with a symmetric union presentation. Let $\overline{V}_L(t) = V_L(t)/(-t^{-1/2} - t^{1/2})^{c-1}$ for an oriented link L, where c is the number of the components of L and $V_L(t)$ is the Jones polynomial of L. (See Section 2 for the definition.)

Theorem 1.1. *Let $\overline{K}$ be a knot with a symmetric union presentation $D_K \cup D_K^*(\infty, m_1, \ldots, m_k)_1$. Then*

$$V_{\overline{K}}(t) = \sum_{(\ell_1,\ldots,\ell_k)} \Pi_{i=1}^k (-t^{-1})^{m_i p_i} (1 - (-t^{-1})^{m_i})^{q_i} \overline{V}_{D_K \cup D_K^*(\infty,\ell_1,\ldots,\ell_k)_w}(t),$$

where $\ell_i \in \{0, \infty\}$,

$$p_i = \begin{cases} 1 & (\ell_i = 0) \\ 0 & (\ell_i = \infty) \end{cases}$$

, $q_i = 1 - p_i$ and $w = \sum_{i=1}^k q_i + 1$ for each k-tuple of tangles $(\ell_1, \ldots, \ell_k)$. The sum is taken over all combinations of k tangles 0's and ∞'s to tangles $\ell_1, \ldots, \ell_k$.

The following corollary generalizes a result of Lamm in the case when $s = 1$ in [5].

Corollary 1.2. *Let $\overline{K}$ be a link with a symmetric union presentation $D_K \cup D_K^*(\infty, m_1, \ldots, m_k)_1$ and s, a positive integer. If $m_i \equiv 0$ mod s for each i, then $V_{\overline{K}}(-\exp(\frac{2\pi i}{s})) = V_K(-\exp(\frac{2\pi i}{s})) \cdot V_{K^*}(-\exp(\frac{2\pi i}{s}))$.*

A knot is called *amphicheiral*, if it is isotopic to its mirror image. By Theorem 1.1, we have the following as special cases.

Theorem 1.3. [7] *Let $\overline{K}$ be a knot with a symmetric union presentation $D_K \cup D_K^*(\infty, m)_1$. Then $t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1}) = (t^m + (-1)^m) V_K(t) \cdot V_K(t^{-1})$. In particular, if $\overline{K}$ is amphicheiral, then $V_{\overline{K}}(t) = V_K(t) V_K(t^{-1})$.*

In this paper, all knots and links are oriented unless otherwise stated. In Section 2, we give the definitions of the Jones polynomial of a link and a symmetric union. In Section 3, we shall prove Theorem 1.1, Corollary 1.2 and Theorem 1.3. In Section 4, we shall consider an amphicheiral symmetric union and the property.

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2 Definitions

Definition 2.1. Let K be a link in $\mathbb{R}^3$. We denote a diagram of K by D_K. The *bracket polynomial* of a diagram of K [3][6], $< D_K >$ can be defined as a polynomial which satisfies the following identities.

i) $< \bigcirc >= 1$,

ii) $< D_K \cup \bigcirc >= -(A^2 + A^{-2}) < D_K >$,

iii) $<$  $>= A <$  $> +A^{-1} <$  $>$.

We define $V_{D_K}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ by $V_{D_K}(t) = \{(-A^3)^{-\delta(D_K)} < D_K >\}_{t^{1/2}=A^{-2}}$ for any diagram D_K for K, where $\delta(D_K)$ is the writhe of D_K. It is shown that $V_{D_K}(t)$ is an invariant of K [3]. Then we denote $V_{D_K}(t)$ by $V_K(t)$ and call it the *Jones polynomial* of K.

Here we give the definition of a symmetirc union as follows. (See [5] for the original definition.) We denote the tangle made of half twists by an integer $m \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ as in Fig. 1.

Definition 2.2. Let D be an unoriented diagram of a knot and D^* the diagram D reflected at an axis in the plane. We replace the tangles $T_i = 0$ $(i = 0, \ldots, k)$ on the symmetry axis by either $T_i = \infty$ or $T_i = m_i \in \mathbb{Z}$ for each i, providing that at least one of T_i's is replaced by ∞ as shown in Fig. 1. We call the result a *symmetric union* and denote it by $D \cup D^*(m_0, \ldots, m_k)_\mu$, where μ is the number of ∞-tangles on the axis.

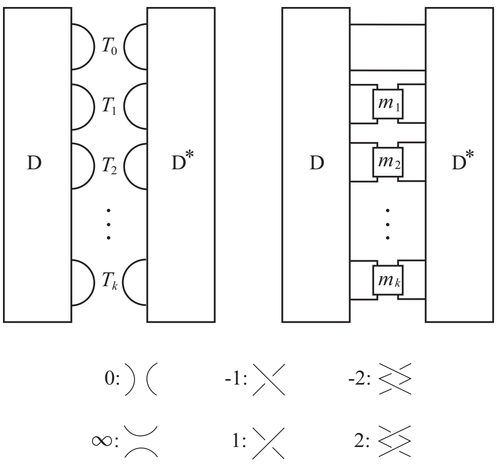

Figure 1: A symmetric union

If a link L has a diagram $D \cup D^*(m_0, \ldots, m_k)_\mu$, then the diagram is called a *symmetric union presentation* for L. We say that a link L is a *symmetric union* if L has a symmetric union presentation.

3 A formula of the Jones polynomial

Let $T_m = \frac{(-A^{-3})^m - A^m}{-A^{-2}-A^2}$ and $S_m = A^m$ for an indeterminate A and a non-zero integer m.

Proof of Theorem 1.1. By using the identities of the bracket polynomial in Defnition 2.1, we have
$< D_K \cup D_K^*(\infty, m_1, m_2, \ldots, m_k)_1 >= A^{m_1/|m_1|} D_K \cup D_K^*(\infty, m_1-1, m_2, \ldots, m_k)_1$
$+F_1 < D_K \cup D_K^*(\infty, \infty, m_2, \ldots, m_k)_2 > = \cdots = A^{\sum_{i=1}^{|m_1|} m_1/|m_1|} < D_K \cup D_K^*(\infty, 0, m_2, \ldots, m_k)_1 >$
$+(F_1 + \cdots + F_{|m_1|}) < D_K \cup D_K^*(\infty, \infty, m_2, \ldots, m_k)_2 > = A^{m_1} < D_K \cup D_K^*(\infty, 0, m_2, \ldots, m_k) >$
$+(\sum_{j=1}^{|m_1|} F_j) < D_K \cup D_K^*(\infty, \infty, m_2, \ldots, m_k)_2 >$, where each F_j is some polynomial.
Now we calculate a formula of $\sum_{j=1}^{|m_1|} F_j$ by considering the unknot instead of K as follows. We assume that D_K is a diagram as in Fig. 2 so that we have a symmetric union of the unknot. We denote the diagram in Fig. 2 by $D_\circ$.

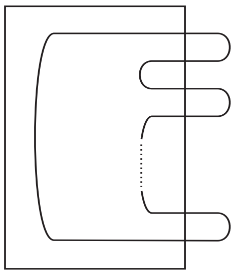

Figure 2: $D_\circ$

Then by calculating each term by using the identities in Definition 2.1, we have
$< D_\circ \cup D_\circ^*(\infty, m_1, m_2, \ldots, m_k)_1 >= (-A^{-3m_1/|m_1|})^{|m_1|} \cdots (-A^{-3m_k/|m_k|})^{|m_k|}(-A^{-2} - A^2)$
$= (-1)^{\sum_{i=1}^k |m_i|} A^{-3\sum_{i=1}^k m_i}(-A^{-2} - A^2)$,
$< D_\circ \cup D_\circ^*(\infty, 0, m_2, \ldots, m_k)_1 >= (-1)^{\sum_{i=2}^k |m_i|} A^{-3\sum_{i=2}^k m_i}(-A^{-2} - A^2)$ and
$< D_\circ \cup D_\circ^*(\infty, \infty, m_2, \ldots, m_k)_2 >= (-1)^{\sum_{i=2}^k |m_i|} A^{-3\sum_{i=2}^k m_i}(-A^{-2} - A^2)^2$.

Then we have
$$\sum_{j=1}^{|m_1|} F_j = \frac{< D_\circ \cup D_\circ^*(\infty, m_1, m_2, \ldots, m_k)_1 > - < D_\circ \cup D_\circ^*(\infty, 0, m_2, \ldots, m_k)_1 >}{< D_\circ \cup D_\circ^*(\infty, \infty, m_2, \ldots, m_k)_2 >}$$
$$= \frac{(-1)^{\sum_{i=1}^k |m_i|} A^{-3\sum_{i=1}^k m_i}(-A^{-2} - A^2) - (-1)^{\sum_{i=2}^k |m_i|} A^{-3\sum_{i=2}^k m_i}(-A^{-2} - A^2)}{(-1)^{\sum_{i=2}^k |m_i|} A^{-3\sum_{i=2}^k m_i}(-A^{-2} - A^2)^2}$$
$$= \frac{(-1)^{|m_1|} A^{-3m_1} - A^{m_1}}{-A^{-2} - A^2} = \frac{(-A^{-3})^{m_1} - A^{m_1}}{-A^{-2} - A^2}.$$
Thus
$< D_K \cup D_K^*(\infty, m_1, m_2, \ldots, m_k)_1 >= A^{m_1} < D_K \cup D_K^*(\infty, 0, m_2, \ldots, m_k)_1 >$
$+\frac{(-A^{-3})^{m_1} - A^{m_1}}{-A^{-2} - A^2} < D_K \cup D_K^*(\infty, \infty, m_2, \ldots, m_k)_2 >$
$= S_{m_1} < D_K \cup D_K^*(\infty, 0, m_2, \ldots, m_k)_1 > +T_{m_1} < D_K \cup D_K^*(\infty, \infty, m_2, \ldots, m_k)_2 >$.

By iterating the same argument, we have
$< D_K \cup D_K^*(\infty, m_1, m_2, \ldots, m_k)_1 >= S_{m_1} S_{m_2} < D_K \cup D_K^*(\infty, 0, 0, m_3, \ldots, m_k)_1 >$
$+S_{m_1} T_{m_2} < D_K \cup D_K^*(\infty, 0, \infty, m_3, \ldots, m_k)_2 >$
$+T_{m_1} S_{m_2} < D_K \cup D_K^*(\infty, \infty, 0, m_3, \ldots, m_k)_2 >$

$+T_{m_1}T_{m_2} < D_K \cup D_K^*(\infty,\infty,\infty,m_3,\ldots,m_k)_3 >$.

Then by induction on k, we know that
$< D_K \cup D_K^*(\infty,m_1,m_2,\ldots,m_k)_1 >= \sum_{(\ell_1,\ldots,\ell_k)} \Pi_{i=1}^k (S_{m_i}^{p_i(\ell_1,\ldots,\ell_k)} T_{m_i}^{q_i(\infty,\ell_1,\ldots,\ell_k)})\times$
$< D_K \cup D_K^*(\infty,\ell_1,\ell_2,\ldots,\ell_k)_w >$, where $\ell_i \in \{0,\infty\}$, $p_i(\ell_1,\ldots,\ell_k) = 1$ or 0 if $\ell_i = 0$ or ∞ respectively, $q_i(\ell_1,\ldots,\ell_k) = 1 - p_i(\ell_1,\ldots,\ell_k)$ and $w = \sum_{i=1}^k q_i(\ell_1,\ldots,\ell_k) + 1$.

Now we obtain that

$$V_{D_K\cup D_K^*(\infty,m_1,\ldots,m_k)_1}(A) = (-A)^{-3\sum_{i=1}^k(-m_i)} < D_K \cup D_K^*(\infty,m_1,m_2,\ldots,m_k)_1 >$$
$$= (-A^3)^{\sum_{i=1}^k m_i} < D_K \cup D_K^*(\infty,m_1,m_2,\ldots,m_k)_1 > = (-A^3)^{\sum_{i=1}^k m_i}\times$$
$$\sum_{(\ell_1,\ldots,\ell_k)} \Pi_{i=1}^k A^{m_ip_i(\ell_1,\ldots,\ell_k)}\Big(\frac{(-A^{-3})^{m_i} - A^{m_i}}{-A^{-2}-A^2}\Big)^{q_i(\ell_1,\ldots,\ell_k)} < D_K \cup D_K^*(\infty,\ell_1,\ldots,\ell_k)_w >$$
$$= \sum_{(\ell_1,\ldots,\ell_k)} \Pi_{i=1}^k (-1)^{m_ip_i(\ell_1,\ldots,\ell_k)} A^{4m_ip_i(\ell_1,\ldots,\ell_k)}\Big(\frac{1-(-1)^{m_i}A^{4m_i}}{-A^{-2}-A^2}\Big)^{q_i(\ell_1,\ldots,\ell_k)} < D_K \cup D_K^*(\infty,\ell_1,\ldots,\ell_k)_w >.$$

Using $t^{1/2} = A^{-2}$, we have

$$V_{D_K\cup D_K^*(\infty,m_1,\ldots,m_k)_1}(t) =$$
$$\sum_{(\ell_1,\ldots,\ell_k)} \Pi_{i=1}^k (-1)^{m_ip_i(\ell_1,\ldots,\ell_k)} t^{-m_ip_i(\ell_1,\ldots,\ell_k)}\Big(\frac{1-(-1)^{m_i}t^{-m_i}}{-t^{1/2}-t^{-1/2}}\Big)^{q_i(\ell_1,\ldots,\ell_k)} V_{D_K\cup D_K^*(\infty,\ell_1,\ldots,\ell_k)_w}(t)$$
$$= \sum_{(\ell_1,\ldots,\ell_k)} \Pi_{i=1}^k (-t^{-1})^{m_ip_i(\ell_1,\ldots,\ell_k)}\Big(\frac{1-(-1)^{m_i}t^{-m_i}}{-t^{1/2}-t^{-1/2}}\Big)^{q_i(\ell_1,\ldots,\ell_k)} V_{D_K\cup D_K^*(\infty,\ell_1,\ldots,\ell_k)_w}(t).$$

Therefore we obtain

$$V_{D_K\cup D_K^*(\infty,m_1,\ldots,m_k)_1}(t) = \sum_{(\ell_1,\ldots,\ell_k)} \Pi_{i=1}^k (-t^{-1})^{m_ip_i(\ell_1,\ldots,\ell_k)}(1-(-1)^{m_i}t^{-m_i})^{q_i(\ell_1,\ldots,\ell_k)}\overline{V}_{D_K\cup D_K^*(\infty,\ell_1,\ldots,\ell_k)_w}(t).$$

Remark 3.1. By a result of Eisermann ([1], Theorem 1), we know that $\overline{V}_{D_K\cup D_K^*(\infty,\ell_1,\ldots,\ell_k)_w}(t)$ in the statement of Theorem 1.1 is always a Laurent polynomial. The integers p_i, q_i and w depend on k-tuple of tangles $(\ell_1,\ldots,\ell_k)$, however we abbreviate the integers for simplicity in the statement of Theorem 1.1.

Proof of Corollary 1.2. By Theorem 1.1, we have $V_{\overline{K}}(-\exp(\frac{2\pi i}{s})) = \overline{V}_{D_K\cup D_K^*(\infty,0,\ldots,0)_1}(-\exp(\frac{2\pi i}{s}))$
$= V_K(-\exp(\frac{2\pi i}{s}))\cdot V_{K^*}(-\exp(\frac{2\pi i}{s}))$.

Proof of Theorem 1.3. In the case when $k=1$ in Theorem 1.1, we have the formula as follows.
$V_{\overline{K}}(t) = (-1)^m t^{-m}\overline{V}_{D_K\cup D_K^*(\infty,0)_1}(t) + (1-(-1)^m t^{-m})\overline{V}_{D_K\cup D_K^*(\infty,\infty)_2}(t)$.
Then the first part of the theorem is obtained as follows.
$t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1}) = \{(-1)^m\overline{V}_{D_K\cup D_K^*(\infty,0)_1}(t) + (t^m-(-1)^m)\overline{V}_{D_K\cup D_K^*(\infty,\infty)_2}(t))\}+$
$\{t^m\overline{V}_{D_K\cup D_K^*(\infty,0)_1}(t) + ((-1)^m - t^m)\overline{V}_{D_K\cup D_K^*(\infty,\infty)_2}(t)\}$
$= (t^m + (-1)^m)\overline{V}_{D_K\cup D_K^*(\infty,0)_1}(t) = (t^m+(-1)^m)V_K(t)V_K(t^{-1})$.
The latter part of the theorem follows immediately from the first part because $V_{\overline{K}}(t) = V_{\overline{K}}(t^{-1})$ if $\overline{K}$ is amphicheiral [6].

4 Examples

For a positive integer m, let K_m be a knot with the symmetric union presentation as described in Fig. 3. It is easily seen that K_m is an amphicheiral knot for each m.

If $m=1$, then the Jones polynomial of K_m is calculated by Theorem 1.1 as follows.

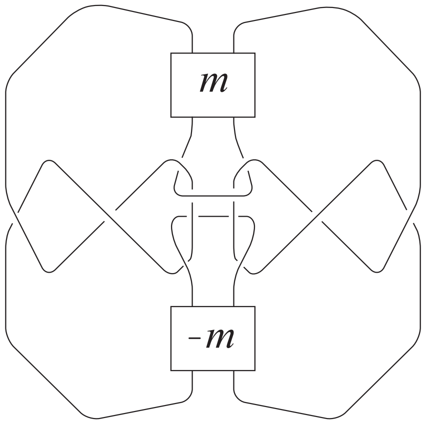

Figure 3: K_m

$V_{K_1}(t) = -t^{-5} + 3t^{-4} - 7t^{-3} + 10t^{-2} - 12t^{-1} + 15 - 12t + 10t^2 - 7t^3 + 3t^4 - t^5$.

By a calculation in [7], we know that $V_t(K_m)$ cannot have a form $f(t)f(t^{-1})$ for any Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$. Thus, by Theorem 1.3, we know that K_1 cannot have a symmetric union presentation $D_K \cup D_K^*(\infty, m)_1$. The Jones polynomials of K_2 and K_3 are calculated as follows.

$V_{K_2}(t) = t^{-7} - 2t^{-6} + 2t^{-5} + t^{-4} - 7t^{-3} + 12t^{-2} - 16t^{-1} + 19 - 16t + 12t^2 - 7t^3 + t^4 + 2t^5 - 2t^6 + t^7$

$V_{K_3}(t) = -t^{-8} + 2t^{-7} - 3t^{-6} + t^{-5} + 5t^{-4} - 14t^{-3} + 24t^{-2} - 31t^{-1} + 35 - 31t + 24t^2 - 14t^3 + 5t^4 + t^5 - 3t^6 + 2t^7 - t^8$

By using Mathematica [8], we know that $V_{K_2}(t)$ and $V_{K_3}(t)$ are irreducible. In particular, they cannot have a form $f(t)f(t^{-1})$ for any Laurent polynomial $f(t)$.

Question. Does $V_{K_m}(t)$ have a form $f(t)f(t^{-1})$ for some Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ for each $m > 1$?

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