1

ON PRIME KNOTS WITH SYMMETRIC UNION PRESENTATIONS

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Abstract. A symmetric union is a knot in the 3-space, obtained from a knot and its mirror image, which are symmetric with respect to a 2-plane in $\mathbb{R}^3$, by taking the connected sum of them and moreover by connecting them with some vertical twists along the plane. In this paper, we give a sufficient condition for a symmetric union to be prime.

Key words: Symmetric union; prime knot.

1. Introduction

A *symmetric union* was originally introduced by Kinoshita and Terasaka [2] and later, Lamm [3] generalized the definition. A symmetric union is known to be an example of a *ribbon knot* [1].

A knot K is *composite* if there is a sphere S intersecting K transversally in two points, such that neither of the 3-balls bounded by S intersects K in a single unknotted spanning arc. The sphere S is called a *decomposing sphere* for L. A non-trivial knot is *prime* if it is not composite. In this paper, we study a prime knot with a symmetric union presentation. In [6], we have given a sufficient condition for a symmetric union to be prime. We refine the result as follows.

Theorem 1.1. *Let K be a symmetric union with minimal twisting number one and $\tilde{D} \cup \tilde{D}^*(m)$, a symmetric union presentation of K. If $\tilde{D} \cup \tilde{D}^*(\infty)$ is a diagram of a trivial link, then K is prime.*

Throughout this paper, $\sharp\{X\}$ denotes the number of elements of X for a finite set X. In Section 2, we shall give the definitions of a symmetric union and the minimal twisting number. In Section 3, we shall prove Theorem 1.1. In Section 4, we shall give some examples.

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2. The definition of a symmetric union

A symmetric union is defined as follows. (See [3] for the original definition.) Let $\mathbb{R}^3$ be the 3-space with x-, y-, and z-axes. Let $\mathbb{R}^3_+ = \{(x, y, z)|x > 0\}$ and $\mathbb{R}^3_- = \{(x, y, z)|x < 0\}$. Throughout this paper, a *tangle* denotes a disjoint union of two arcs properly embedded in a 3-ball. We denote the tangle made of $|m|$ half-twists along the z-axis as a diagram by an integer $m \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ with respect to the x-axis as in Figure 1.

Definition 2.1. We take a knot $\tilde{K}$ in $\mathbb{R}^3_-$ and its mirror image $\tilde{K}^*$ in $\mathbb{R}^3_+$ such that $\tilde{K}$ and $\tilde{K}^*$ are symmetric with respect to the yz-plane $\mathbb{R}^2_{yz}$ as in Figure 2(a). Here we consider a diagram of a knot in the xz-plane $\mathbb{R}^2_{xz}$ and we denote the diagrams of $\tilde{K}$ and $\tilde{K}^*$ by $\tilde{D}$ and $\tilde{D}^*$, respectively. Each disk-arc pair of $T_0, T_1, \ldots, T_k$ as in Figure 2(a) denotes a diagram of the tangle 0. Then we replace the tangles $T_0, T_1, \ldots, T_k$ with tangles ∞, $m_1, m_2, \ldots, m_k$ as in Figure 2(b). (See Figure 3 for example.) Here we

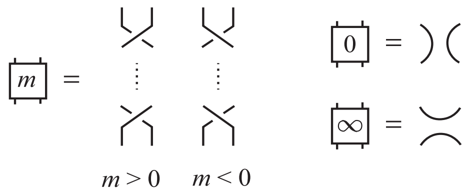

FIGURE 1

assume that $m_i \neq \infty$ $(1 \leq i \leq k)$. The resultant diagram is called a *symmetric union presentation* and we denote it by $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$.

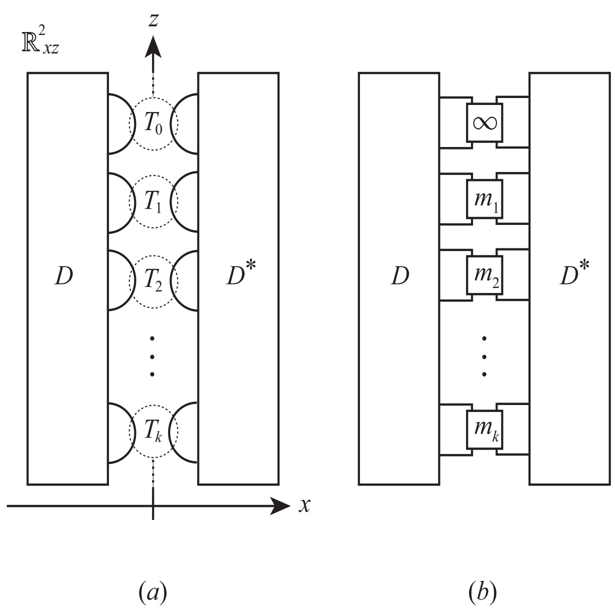

FIGURE 2

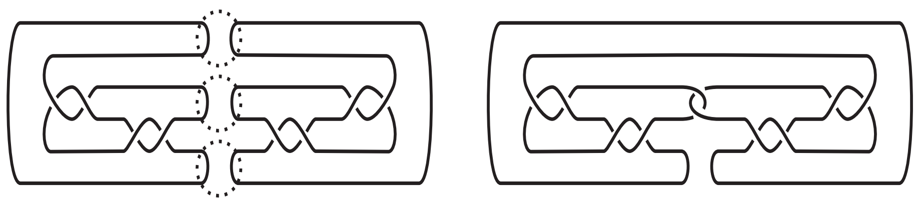

FIGURE 3

If a knot K has a diagram $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$, then the knot K is called a *symmetric union*.

Here we define the minimal twisting number for a symmetric union which was originally introduced in [5] as follows.

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Definition 2.2. We call the number of non-zero elements in $\{m_1, \dots, m_k\}$ the *twisting number* for $\tilde{D} \cup \tilde{D}^*(m_1, \dots, m_k)$. The *minimal twisting number* of a symmetric union K is the smallest number of the twisting numbers of all symmetric union presentations to K.

Remark 2.3. The minimal twisting number is an invariant of K. The minimal twisting number of a two-bridge symmetric union is equal to either one or two [4]. We gave an example of a symmetric union with minimal twisting number two in [5].

3. Proof of Theorem 1.1

Proof of Theorem 1.1. In the case when m is an odd number, the primeness of K follows from the proof of Proposition 6.1 [6]. From now on, we use notations defined in [6]. We assume that K is composite in the case when m is an even number. Then it is enough to consider the case when $\sharp\{d_g \cap K\} = 2$ in the proof of Proposition 6.1 [6]. In that case, we have the following two cases:
(i) $d_t \subset B$; or
(ii) $d_t \subset \overline{\mathbb{B}}^3_{\pm}$.
In the case of (i), we have a connected summand of a trivial knot if $\sharp\{d_t \cap K\} = 2$ and $m = 0$ if $d_t \cap K = \emptyset$. This is a contradiction. In the case of (ii), as in the proof of Theorem 1.2 [6], we take a simple arc α on d_t which connects two intersection points of K and d_t. Then we have a 2-component (split) link L by a surgery of K along α. Then it is easily seen that the linking number of L is non-zero since m is non-zero. This is a contradiction. Thus we know that K is a prime knot.

4. Symmetric unions of two-bridge knots

Example 4.1. We consider symmetric unions of two-bridge knots. We denote the following two-bridge knot by $T(b_1, b_2, \dots, b_{2d})$ $(d > 0)$. (Each b_i $(1 \le i \le 2d)$ denotes the number of full-twists.)

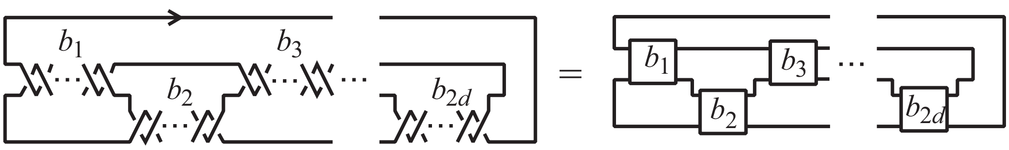

Figure 4. A 2-bridge knot

We take a symmetric union of $T(b_1, b_2, \dots, b_{2d})$ and its mirror image, as in Figure 5. We denote the resulting knot by $B_m(b_1, b_2, \dots, b_{2d})$. In Figure 5, m represents vertically arranged $|m|$ crossings, which are right-handed if $m > 0$ and left-handed if $m < 0$.

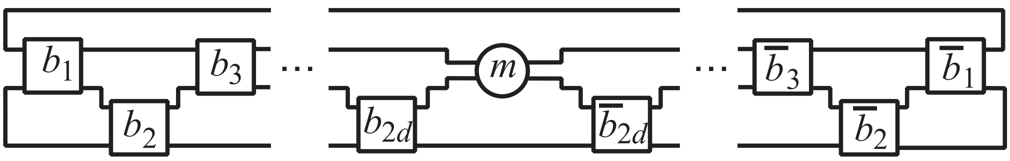

Figure 5. A symmetric union of 2-bridge knots

It is easily seen that $B_\infty(b_1, b_2, \dots, b_{2d})$ is a trivial link. By using the formula of the Jones polynomial in [5], we know that the Jones polynomial of $B_m(b_1, b_2, \dots, b_{2d})$ is as follows:
$V(t) = (-1)^m t^{-m} F(t)F(t^{-1}) + ((-1)^m - 1)t^{-m}$, where $F(t)$ is the Jones polynomial of $T(b_1, b_2, \dots, b_{2d})$. Since $T(b_1, b_2, \dots, b_{2d})$ is the alternating knot, we know that the reduced degree of the Jones polynomial is $\Sigma_{i=1}^{2d} d_i$ and then the minimal degree of $F(t)F(t^{-1})$ is $-\Sigma_{i=1}^{2d} d_i$. If $-m > \Sigma_{i=1}^{2d} d_i \ge 2$, then the minimal

degree of $V(t)$ is $-\Sigma_{i=1}^{2d} d_i - m > 0$. In particular, $B_m(b_1, b_2, \ldots, b_{2d})$ is not amphicheiral and so the minimal twisting number of $B_m(b_1, b_2, \ldots, b_{2d})$ is non-zero. Thus we know that $B_m(b_1, b_2, \ldots, b_{2d})$ is a prime knot if $-m > \Sigma_{i=1}^{2d} d_i \geq 2$ by Theorem 1.1. We also find that the maximal degree of the Jones polynomial of $B_m(b_1, b_2, \ldots, b_{2d})$ is $\Sigma_{i=1}^{2d} d_i - m$ if $-m > \Sigma_{i=1}^{2d} d_i \geq 2$. Then by Corollary 7.2 [6], we know that the minimal twisting number of $B_{m_1}(b_1, b_2, \ldots, b_{2d}) \sharp B_{m_2}(b_1, b_2, \ldots, b_{2d})$ is two if $-m_1 > \Sigma_{i=1}^{2d} d_i \geq 2$ and $-m_2 > \Sigma_{i=1}^{2d} d_i \geq 2$, where $K_1 \sharp K_2$ denotes the connected sum of two knots K_1 and K_2.

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