岐阜大学教育学部研究報告（自然科学）第42巻，9-13，2018年３月
Sci. Rep. Fac. Educ. Gifu Univ.（Nat. Sci.）42, 9-13, March 2018

ON THE NUMBER OF p-COLORINGS OF KNOTS

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ABSTRACT. We show some examples of pairs of knots with the same number of p-colorings for any positive integer $p \geq 2$. In particular, we show that there exists an infinite family of knots with only trivial p-colorings.

Key words: Knots; coloring; congruence.

1. INTRODUCTION

In the 1950's, Fox introduced the concept of a *p-coloring* of a knot, where $p \geq 2$ is an integer. A *p-coloring* of a diagram D of a knot K is a map $f : \{\text{arcs of } D\} \to \mathbb{Z}/p\mathbb{Z}$ such that at each crossing, if the three arcs are mapped to x, y and z, with the z corresponding to the overcrossing, then they satisfy $x + y - 2z \equiv 0 \pmod{p}$. We call the number of distinct p-colorings of a diagram of a knot K, the *rank* of the p-colorings of K and we denote it by $\mathrm{col}_p(K)$. We say that a p-coloring is *trivial* if it is a constant map. A diagram for K is *p-colorable* if it has a non-trivial p-coloring. The rank of the p-colorings of a knot and the p-colorabiliy are invariants of a knot. We know that if a knot is p-colorable, then it is also pq-colorable for any positive integer q. Conversely, if a knot is pq-colorable for some positive integers p and q, then the knot is either p-colorable or q-colorable [5].

In this paper, we obtain the following results.

Theorem 1.1. *Each pair of knots in the set* $\{(4_1, 5_1),\ (6_2, 7_2),\ (7_4, 3_1\sharp 4_1)\}$ (*Figure* 1) *is knots with the same rank of the p-colorings for any positive integer* $p \geq 2$.

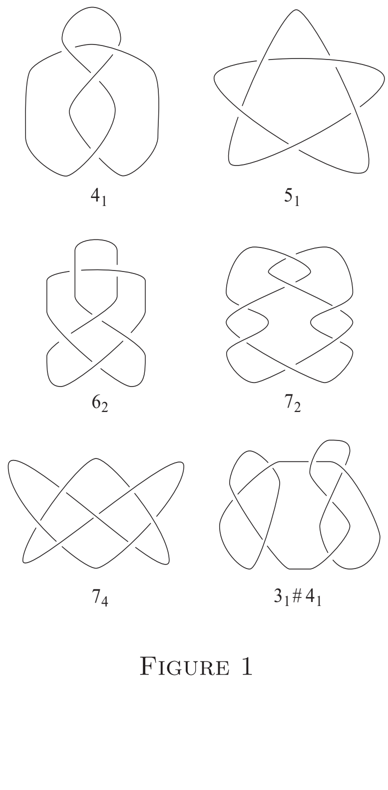

FIGURE 1

In particular,

$$\mathrm{col}_p(4_1) = \begin{cases} 5p & (p = 5m \ (m \in \mathbb{N})) \\ p & (p \neq 5m \ (m \in \mathbb{N})) \end{cases} \tag{1}$$

$$\mathrm{col}_p(6_2) = \begin{cases} 11p & (p = 11m \ (m \in \mathbb{N})) \\ p & (p \neq 11m \ (m \in \mathbb{N})) \end{cases} \tag{2}$$

$$\mathrm{col}_p(7_4) = \begin{cases} 15p & (p = 15n \ (n \in \mathbb{N})) \\ 3p & (p = 3m, p \neq 5n \ (m, n \in \mathbb{N})) \\ 5p & (p \neq 3m, p = 5n \ (m, n \in \mathbb{N})) \\ p & (p \neq 3m, 5n \ (m, n \in \mathbb{N})) \end{cases} . \tag{3}$$

Theorem 1.2. *There exists an infinite family of knots* $\{K_i\}$ *such that* $\mathrm{col}_p(K_i) = p$ *for any positive integer* $p \geq 2$.

In Sections 2, we shall give proofs of Theorems 1.1 and 1.2.

Acknowledgements. The second author is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research(C), 2016-2018 (16K05145).

2. Proofs

Proof of Theorem 1.1. First we take a p-coloring of 4_1 as in Figure 2.

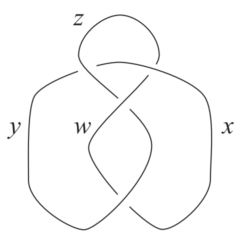

Figure 2

Then we have the following congruences:
$2z \equiv x + y \pmod p$
$2x \equiv z + w \pmod p$
$2w \equiv z + y \pmod p$
$2y \equiv w + x \pmod p$
Thus we have the following congruences:
$x + y - 2z \equiv 0 \pmod p$
$-5z + 5w \equiv 0 \pmod p$
$y + z - 2w \equiv 0 \pmod p$

In the case when $p \neq 5m$, we have $z \equiv w$ by the second congruence. Thus we have $x \equiv z$ and $y \equiv z$ by the first and the third congruences. Therefore we know that the rank is p.

In the case when $p = 5m$, we have $-z + w \equiv 0 \pmod m$ by the second congruence. Then we have $25m$ choices for the pair (w, z). Thus we know that the rank is $25m$.

Next we consider a p-coloring of 5_1 as in Figure 3.
Then we have the following congruences:
$2v \equiv x + y \pmod p$
$2x \equiv z + v \pmod p$
$2z \equiv w + x \pmod p$
$2w \equiv z + y \pmod p$

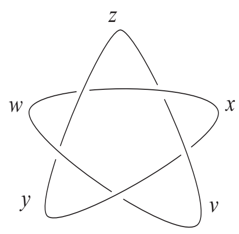

FIGURE 3

$2y \equiv v + w \pmod p$
Thus we have the following congruences:
$x + 3y - 4z \equiv 0 \pmod p$
$5y - 5z \equiv 0 \pmod p$
$w - 3y + 2z \equiv 0 \pmod p$
$v + y - 2z \equiv 0 \pmod p$

In the case when $p \neq 5m$, we have $y \equiv z$ by the second congruence. Thus we have $x \equiv z$, $w \equiv y$ and $v \equiv z$ by the first, the third and the fourth congruences. Therefore we know that the rank is p.

In the case when $p = 5m$, we have $y - z \equiv 0 \pmod m$ by the second congruence. Then we have $25m$ choices for the pair (y, z). Thus we know that the rank is $25m$.

Thus we know that the ranks of the p-colorings of 4_1 and 5_1 are equal. By the same way, we can show that 6_2 and 7_2 have the same rank of the p-colorings.

We consider a p-coloring of 7_4 as in Figure 3.

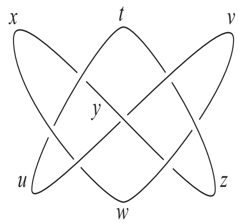

FIGURE 4

Then we have the following congruences:
$2t \equiv x + y \pmod p$
$2x \equiv u + t \pmod p$
$2u \equiv x + w \pmod p$
$2y \equiv u + v \pmod p$
$2v \equiv t + z \pmod p$
$2z \equiv v + w \pmod p$
$2w \equiv y + z \pmod p$
Thus we have the following congruences:
$-5x + 5z \equiv 0 \pmod p$
$-w - 2x + 2y + z \equiv 0 \pmod p$
$-15x + 15y \equiv 0 \pmod p$
$u + 2x - 4y + z \equiv 0 \pmod p$
$t - 4x + 4y - z \equiv 0 \pmod p$
$-4x + 3y + z \equiv 0 \pmod p$

In the case when $p \neq 3m, 5n$, we know that $x \equiv y \equiv z$ by the first and the third congruences. Thus we have $x \equiv t \equiv u \equiv v \equiv w$. Hence the rank is equal to p.

In the case when $p = 5n$ and $p \neq 3m$, we have $x \equiv z \pmod{n}$ by the first congruence. Thus we have $25n$ choices for the pair (x, z). We also know that $x \equiv y \pmod{p}$ by the assumption and the third congruence. Hence we know that the rank is equal to $25n$ by the rest congruences.

In the case when $p = 3m$ and $p \neq 5n$, we have $5x - 5z \equiv 0 \pmod{m}$ where m is not divisible by 5. Thus we have $x \equiv z \pmod{m}$ and we have $9m$ choices for the pair (x, z). Hence the rank is equal to $9m$ by the rest congruences.

In the case when $p = 15n$, we know that $x \equiv y \pmod{n}$ by the third congruence. Thus we have $225n$ choices for the pair (x, y). Hence we know that the rank is equal to $225n$ by the rest congruence.

We consider a p-coloring of $3_1 \sharp 4_1$ as in Figure 3.

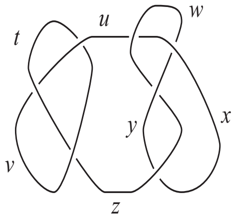

Figure 5

Then we have the following congruences:
$2t \equiv u + v \pmod{p}$
$2u \equiv t + v \pmod{p}$
$2v \equiv t + z \pmod{p}$
$2w \equiv x + u \pmod{p}$
$2x \equiv y + w \pmod{p}$
$2y \equiv w + z \pmod{p}$
$2z \equiv x + y \pmod{p}$
Then we have the following congruences:
$5y - 5z \equiv 0 \pmod{p}$
$3v - 3z \equiv 0 \pmod{p}$
$x + v - 2z \equiv 0 \pmod{p}$
$u - z \equiv 0 \pmod{p}$
$w - 2y + z \equiv 0 \pmod{p}$
$x + y - 2z \equiv 0 \pmod{p}$

In the case when $p \neq 3m, 5n$, we know that $v \equiv y \equiv z$ by the first and the second congruences. Thus we have $v \equiv t \equiv u \equiv w \equiv x$. Hence the rank is equal to p.

In the case when $p = 5n$ and $p \neq 3m$, we have $y \equiv z \pmod{n}$ by the first congruence. Thus we have $25n$ choices for the pair (y, z). We also know that $v \equiv y \pmod{p}$ by the assumption and the second congruence. Hence we know that the rank is equal to $25n$ by the rest congruences.

In the case when $p = 3m$ and $p \neq 5n$, we have $v \equiv z \pmod{m}$ by the second congruence. Thus we have $9m$ choices for the pair (v, z). We also know that $y \equiv z \pmod{p}$ by the assumption and the first congruence. Hence the rank is equal to $9m$ by the rest congruences.

In the case when $p \neq 15n$, we know that $y \equiv z \pmod{5n}$ by the first congruence. We also know that $v \equiv z \pmod{3n}$ by the second congruence. Thus we have $225n$ choices for the triad (v, y, z). Hence we know that the rank is equal to $225n$ by the rest congruence.

Proof of Theorem 1.2. Let K_c be a knot as in Figure 6. (The number c denotes the number of full-twists.) We know that $\det(K_c)$=1 [3], where $\det(K_c)$ is the *knot determinant* of K_c . If p is a prime number, then the knot K_c is not p-colorable since p cannot divide $\det(K_c)$ [1]. Suppose that K_c is p-colorable for some positive integer $p \geq 2$. Then K_c is d-colorable for some prime number d [5], where d is a prime factor of p. This is a contradiction. Thus we know that K_c is not p-colorable for any positive integer $p \geq 2$. Hence $\mathrm{col}_p(K_c) = p$ for any positive integer $p \geq 2$.

The *Jones polynomial* $V(t)$ of K_c is easily calculated as follows. (See [2] for the definition. We may use [Theorem 1.1, [4]] for the calculation.)

$V(t) = (-1)^c t^{-c} + (1 - (-1)^c t^{-c})(-t^4 + 2t^3 - 3t^2 + 4t - 3 + 4t^{-1} - 3t^{-2} + 2t^{-3} - t^{-4})$.

Then we know that K_{c_1} is not equivalent to K_{c_2} if $c_1 \neq c_2$. This completes the proof.

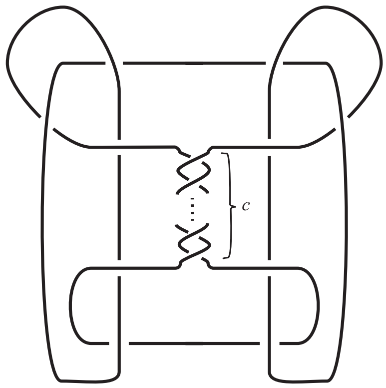

Figure 6

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