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Littlewood-Paley and Lusin functions of α-parabolic type

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ABSTRACT. For $0 < \alpha \leq 1$, we consider the $L^{(\alpha)}$-harmonic extensions of L^2-functions on the Euclidean space $\mathbb{R}^n$. In this paper, we study Littlewood-Paley and Lusin functions for $L^{(\alpha)}$-harmonic extensions, and we give some identities concerning L^2-norms of thier functions.

1. Introduction

Let $n \geq 1$ and H be the upper half-space of the $(n+1)$-dimensional Euclidean space, that is, $H = \{X = (x, t) \in \mathbb{R}^{n+1} : x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

(1.1) $$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha,$$

where $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $\Delta_x = \partial_1^2 + \cdots + \partial_n^2$. Let $C(H)$ be the set of all real-valued continuous functions on H. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions (for details, see Section 2). In this paper, we study Littlewood-Paley and Lusin functions for $L^{(\alpha)}$-harmonic extensions, and we give some identities concerning L^2-norms of thier functions.

To state our main results, we give some definitions. For $1 \leq p \leq \infty$, the Lebesgue space $L^p := L^p(\mathbb{R}^n, dV_n)$ is defined to be the Banach space of Lebesgue measurable (real-valued) functions on $\mathbb{R}^n$ with norm $\| \cdot \|_{L^p}$, where dV_n is the Lebesgue measure on $\mathbb{R}^n$. We denote by $W^{(\alpha)}$ the fundamental solution of $L^{(\alpha)}$ (see Section 2 for the definition). We define an $L^{(\alpha)}$-harmonic extension $\mathcal{H}_f^{(\alpha)}$ of $f \in L^p$ by

(1.2) $$\mathcal{H}_f^{(\alpha)}(x, t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x - y, t) f(y) dV_n(y), \qquad (x, t) \in H.$$

It is shown that the function $\mathcal{H}_f^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H (see [4, Theorem 5.2]). It is well known that when $\alpha = 1/2$, the fundamental solution $W^{(1/2)}$ coincides with the Poisson kernel for H (see [5, Section 2]). Therefore, the function $\mathcal{H}_f^{(1/2)}$ is the usual harmonic extensions of f.

For a real number κ, let $\mathcal{D}_t^\kappa = (-\partial_t)^\kappa$ be a fractional differential operator, and $\mathcal{FC}^\kappa$ the class of functions φ on $\mathbb{R}_+ = (0, \infty)$ such that $\mathcal{D}_t^\kappa \varphi$ is well-defined (for the explicit definitions of $\mathcal{D}_t^\kappa$

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and $\mathcal{FC}^{\kappa}$, see Section 2). For a function u on H, let $\nabla_x u = (\partial_1 u, \cdots, \partial_n u)$ and $|\nabla_x u(x,t)|^2 = \sum_{j=1}^{n} |\partial_j u(x,t)|^2$. Furthermore, let Γ be the gamma function.

In this paper, we show the following theorem, which are identities of Littlewood-Paley type for $L^{(\alpha)}$-harmonic extensions. When $\alpha = 1/2$, the following identities are well known (see [7, pp. 82–83]).

THEOREM 1. *Let* $0 < \alpha \leq 1$ *and* $f \in L^2$. *Then the following identities hold*:

$$\int_0^{\infty} \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1} |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt = 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2$$

and

$$\int_0^{\infty} \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1} |\nabla_x \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt = 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2.$$

We also show the following theorem, which are identities of Lusin type for $L^{(\alpha)}$-harmonic extensions. When $\alpha = 1/2$, the following identities are well known (see [6]).

For $\xi \in \mathbb{R}^n$ and $\rho > 0$, let

$$C_\rho^{(\alpha)}(\xi) := \{(x,t) \in H : |x - \xi|^{2\alpha} \leq \rho^{-1} t\}.$$

We define Lusin functions for $L^{(\alpha)}$-harmonic extensions. Let

$$\mathcal{S}_{f,t}^{(\alpha)}(\xi) = \left(\iint_{C_\rho^{(\alpha)}(\xi)} t^{\frac{1}{\alpha}-1-\frac{n}{2\alpha}} |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt \right)^{1/2}$$

and

$$\mathcal{S}_{f,x}^{(\alpha)}(\xi) = \left(\iint_{C_\rho^{(\alpha)}(\xi)} t^{\frac{1}{\alpha}-1-\frac{n}{2\alpha}} |\nabla_x \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt \right)^{1/2}.$$

THEOREM 2. *Let* $0 < \alpha \leq 1$ *and* $f \in L^2$. *Furthermore, let* d_n *be the volume of the unit ball of* $\mathbb{R}^n$. *Then the following identities hold*:

$$\int_{\mathbb{R}^n} |\mathcal{S}_{f,t}^{(\alpha)}(\xi)|^2 dV_n(\xi) = d_n \rho^{-\frac{n}{2\alpha}} 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2$$

and

$$\int_{\mathbb{R}^n} |\mathcal{S}_{f,x}^{(\alpha)}(\xi)|^2 dV_n(\xi) = d_n \rho^{-\frac{n}{2\alpha}} 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2.$$

We describe the construction of this paper. In Section 2, we recall definitions of the $L^{(\alpha)}$-harmonic functions and fractional differential operators. In Section 3, we show the identities of Littlewood-Paley type in Theorem 1 (see Theorem 3.3 in Section 3). In Section 4, we also show the identities of Lusin type in Theorem 2 (see Theorem 4.1 in Section 4).

2. Preliminaries

In this section, we recall some basic properties. We begin with describing the operator $(-\Delta_x)^\alpha$ and the $L^{(\alpha)}$-harmonic functions. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C^\infty(H)$ denote the set of all infinitely differentiable functions on H. Furthermore, let $C_c^\infty(H)$ be the set of all functions in $C^\infty(H)$ with compact support. For $0 < \alpha < 1$, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x,t) := -C_{n,\alpha} \lim_{\varepsilon \to +0} \int_{|y|>\varepsilon} \frac{\psi(x+y,t) - \psi(x,t)}{|y|^{n+2\alpha}} dV_n(y) \tag{2.1}$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma\big((n+2\alpha)/2\big)/\Gamma(-\alpha) > 0$. Let $\widetilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)} u = 0$ in the sense of distributions, that is,

$$\int_H |u\, \widetilde{L}^{(\alpha)} \psi| dV_{n+1} < \infty \quad \text{and} \quad \int_H u\, \widetilde{L}^{(\alpha)} \psi dV_{n+1} = 0 \text{ for all } \psi \in C_c^\infty(H).$$

We describe the fundamental solution of $L^{(\alpha)}$. For $(x,t) \in H$, let

$$\begin{aligned} W^{(\alpha)}(x,t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x \cdot \xi)\, dV_n(\xi) \\ &= \int_{\mathbb{R}^n} e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i x \cdot \xi}\, dV_n(\xi). \end{aligned} \tag{2.2}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and it is $L^{(\alpha)}$-harmonic on H. Furthermore, $W^{(\alpha)} \in C^\infty(H)$.

We also recall definitions of the fractional integral and differential operators for functions on $\mathbb{R}_+ = (0,\infty)$ (for details, see [2]). For a real number $\kappa > 0$, let

$$\mathcal{FC}^{-\kappa} := \big\{\varphi \in C(\mathbb{R}_+) : \varphi(t) = O(t^{-\kappa'})\ (t \to \infty) \text{ for some } \kappa' > \kappa\big\}. \tag{2.3}$$

For a function $\varphi \in \mathcal{FC}^{-\kappa}$, we can define the fractional integral $\mathcal{D}_t^{-\kappa}\varphi$ of φ by

$$\mathcal{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \varphi(\tau + t) d\tau, \qquad t \in \mathbb{R}_+. \tag{2.4}$$

We put $\mathcal{FC}^0 := C(\mathbb{R}_+)$ and $\mathcal{D}_t^0 \varphi := \varphi$. Moreover, let

$$\mathcal{FC}^\kappa := \{\varphi\ ;\ \partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil - \kappa)}\}, \tag{2.5}$$

where $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. Then, we can also define the fractional derivative $\mathcal{D}_t^\kappa \varphi$ of $\varphi \in \mathcal{FC}^\kappa$ by

$$\mathcal{D}_t^\kappa \varphi(t) := \mathcal{D}_t^{-(\lceil\kappa\rceil-\kappa)} \left((-\partial_t)^{\lceil\kappa\rceil}\varphi\right)(t), \qquad t \in \mathbb{R}_+. \tag{2.6}$$

Clearly, when $\kappa \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the operator $\mathcal{D}_t^\kappa$ coincides with the ordinary differential operator $(-\partial_t)^\kappa$. For a real number κ, we may call both (2.4) and (2.6) *the fractional derivatives* *of φ with order κ.* And, we call $\mathcal{D}_t^\kappa$ *the fractional differential operator with order κ.* Here, we give some examples of fractional derivatives of elementary functions.

EXAMPLE 2.1. *Let $\kappa > 0$ and ν be real numbers. Then, we have the following.*

(1) $\mathcal{D}_t^\nu e^{-\kappa t} = \kappa^\nu e^{-\kappa t}$.

(2) *If $-\kappa < \nu$, then* $\mathcal{D}_t^\nu t^{-\kappa} = \dfrac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} t^{-\kappa-\nu}$.

3. Littlewood-Paley functions of α-parabolic type

For a function $f \in L^2$, we denote by $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f, that is,

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i \xi \cdot y} \, dV_n(y), \qquad \xi \in \mathbb{R}^n.$$

Let $n \geq 1$ and $0 < \alpha \leq 1$ be fixed. For $\gamma \in \mathbb{N}_0^n$ and $1 \leq p \leq \infty$, define the interval $I(\gamma, p)$ by

$$I(\gamma,p) := \begin{cases} \{\nu \in \mathbb{R} : \nu > -(n/2\alpha)(1/p) - |\gamma|/2\alpha\} & (p \neq \infty) \\ \{\nu \in \mathbb{R} : \nu > -|\gamma|/2\alpha\} \cup \{0\} & (p = \infty). \end{cases}$$

LEMMA 3.1. ([3, Theorem 3.4]) *Let $0 < \alpha \leq 1$, $1 \leq p \leq \infty$, and $\gamma \in \mathbb{N}_0^n$. If $f \in L^p$ and $\nu \in I(\gamma, p)$, then the derivative $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)$ is well defined, and*

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t) f(y) dV_n(y).$$

Furthermore, there exists a constant $C = C(n, \alpha, p, \gamma, \nu) > 0$ such that

$$|\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)| \leq C t^{-(n/2\alpha)(1/p) - |\gamma|/2\alpha - \nu} \|f\|_{L^p}$$

for all $(x,t) \in H$.

We give properties of fractional derivatives of $L^{(\alpha)}$-harmonic extensions.

LEMMA 3.2. *Let $0 < \alpha \leq 1$ and $f \in L^2$. Then the following statements hold*:

(1) *For a real number $\nu > -\frac{n}{2\alpha}$,*

$$\mathcal{D}_t^\nu W^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} |2\pi\xi|^{2\alpha\nu} e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i x \cdot \xi} \, dV_n(\xi).$$

Furthermore, for integers $1 \leq j \leq n$ and $\ell \in \mathbb{N}_0$,

$$\partial_j^\ell W^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} (2\pi i \xi_j)^\ell e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i x \cdot \xi} \, dV_n(\xi).$$

(2) *For a real number* $\nu > -\frac{n}{4\alpha}$,

$$\mathcal{D}_t^{\nu}\mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} |2\pi\xi|^{2\alpha\nu}\hat{f}(\xi)e^{-t|2\pi\xi|^{2\alpha}}e^{2\pi ix\cdot\xi}\,dV_n(\xi).$$

Furthermore, for integers $1 \le j \le n$ *and* $\ell \in \mathbb{N}_0$,

$$\partial_j^{\ell}\mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} (2\pi i\xi_j)^{\ell}\hat{f}(\xi)e^{-t|2\pi\xi|^{2\alpha}}e^{2\pi ix\cdot\xi}\,dV_n(\xi).$$

PROOF. (1) Since $\nu > -\frac{n}{2\alpha}$, we have

$$\int_0^{\infty}\int_{\mathbb{R}^n} \tau^{\lceil\nu\rceil-\nu-1}|2\pi\xi|^{2\alpha\lceil\nu\rceil}e^{-(\tau+t)|2\pi\xi|^{2\alpha}}\,dV_n(\xi)\,d\tau < \infty.$$

Differentiating through the integral (2.2) with respect to t, the Fubini theorem and Example 2.1 (1) imply that

$$\begin{aligned}
\mathcal{D}_t^{\nu}W^{(\alpha)}(x,t) &= \frac{1}{\Gamma(\lceil\nu\rceil-\nu)}\int_0^{\infty}\tau^{\lceil\nu\rceil-\nu-1}\int_{\mathbb{R}^n}\mathcal{D}_t^{\lceil\nu\rceil}e^{-(\tau+t)|2\pi\xi|^{2\alpha}}e^{2\pi ix\cdot\xi}\,dV_n(\xi)\,d\tau \\
&= \int_{\mathbb{R}^n}\left(\frac{1}{\Gamma(\lceil\nu\rceil-\nu)}\int_0^{\infty}\tau^{\lceil\nu\rceil-\nu-1}\mathcal{D}_t^{\lceil\nu\rceil}e^{-(\tau+t)|2\pi\xi|^{2\alpha}}\,d\tau\right)e^{2\pi ix\cdot\xi}\,dV_n(\xi) \\
&= \int_{\mathbb{R}^n}\left(\mathcal{D}_t^{\nu}e^{-t|2\pi\xi|^{2\alpha}}\right)e^{2\pi ix\cdot\xi}\,d\tau\,dV_n(\xi) \\
&= \int_{\mathbb{R}^n}|2\pi\xi|^{2\alpha\nu}e^{-t|2\pi\xi|^{2\alpha}}e^{2\pi ix\cdot\xi}\,dV_n(\xi).
\end{aligned}$$

Furthermore, differentiating through the integral (2.2) with respect to x, we have

$$\partial_j^{\ell}W^{(\alpha)}(x,t) = \int_{\mathbb{R}^n}(2\pi i\xi_j)^{\ell}e^{-t|2\pi\xi|^{2\alpha}}e^{2\pi ix\cdot\xi}\,dV_n(\xi).$$

(2) By Lemma 3.1 and Lemma 3.2 (1), we have

$$\begin{aligned}
\mathcal{D}_t^{\nu}\mathcal{H}_f^{(\alpha)}(x,t) &= \int_{\mathbb{R}^n}\mathcal{D}_t^{\nu}W^{(\alpha)}(x-y,t)f(y)dV_n(y) \\
&= \int_{\mathbb{R}^n}f(y)\int_{\mathbb{R}^n}|2\pi\xi|^{2\alpha\nu}e^{-t|2\pi\xi|^{2\alpha}}e^{2\pi i(x-y)\cdot\xi}\,dV_n(\xi)\,dV_n(y) \\
&= \int_{\mathbb{R}^n}|2\pi\xi|^{2\alpha\nu}\left(\int_{\mathbb{R}^n}f(y)e^{-2\pi iy\cdot\xi}\,dV_n(y)\right)e^{-t|2\pi\xi|^{2\alpha}}e^{2\pi ix\cdot\xi}\,dV_n(\xi) \\
&= \int_{\mathbb{R}^n}|2\pi\xi|^{2\alpha\nu}\hat{f}(\xi)e^{-t|2\pi\xi|^{2\alpha}}e^{2\pi ix\cdot\xi}\,dV_n(\xi).
\end{aligned}$$

Furthermore, we have

$$\partial_j^{\ell}\mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n}\partial_j^{\ell}W^{(\alpha)}(x-y,t)f(y)dV_n(y)$$

$$= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} (2\pi i \xi_j)^\ell e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i(x-y)\cdot\xi} \, dV_n(\xi) \, dV_n(y)$$
$$= \int_{\mathbb{R}^n} (2\pi i \xi_j)^\ell \left(\int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi} \, dV_n(y) \right) e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i x\cdot\xi} \, dV_n(\xi)$$
$$= \int_{\mathbb{R}^n} (2\pi i \xi_j)^\ell \hat{f}(\xi) e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i x\cdot\xi} \, dV_n(\xi).$$

This completes the proof. □

We give identities of Littlewood-Paley type for $L^{(\alpha)}$-harmonic extensions.

THEOREM 3.3. *Let* $0 < \alpha \le 1$ *and* $f \in L^2$. *Then the following identities hold*:

$$\int_0^\infty \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1} |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x) \, dt = 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2 \tag{3.1}$$

$$\int_0^\infty \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1} |\nabla_x \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x) \, dt = 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2. \tag{3.2}$$

PROOF. We show the identity (3.1). By Lemma 3.2 (2), we have

$$\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} |2\pi\xi| \hat{f}(\xi) e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i x\cdot\xi} \, dV_n(\xi) = \mathcal{F}^{-1}(\varphi_t)(x),$$

where $\varphi_t(\xi) = |2\pi\xi| \hat{f}(\xi) e^{-t|2\pi\xi|^{2\alpha}}$. Therefore, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1} |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x) \, dt = \int_0^\infty t^{\frac{1}{\alpha}-1} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}(\varphi_t)(x)|^2 dV_n(x) \, dt$$
$$= \int_0^\infty t^{\frac{1}{\alpha}-1} \int_{\mathbb{R}^n} |\varphi_t(\xi)|^2 dV_n(\xi) \, dt = \int_0^\infty t^{\frac{1}{\alpha}-1} \int_{\mathbb{R}^n} |2\pi\xi|^2 |\hat{f}(\xi)|^2 e^{-2t|2\pi\xi|^{2\alpha}} dV_n(\xi) \, dt$$
$$= \int_{\mathbb{R}^n} |2\pi\xi|^2 |\hat{f}(\xi)|^2 \int_0^\infty t^{\frac{1}{\alpha}-1} e^{-2t|2\pi\xi|^{2\alpha}} dt \, dV_n(\xi) = 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 dV_n(\xi).$$

We show the identity (3.2). By Lemma 3.2 (2), for $1 \le j \le n$, we have

$$\partial_j \mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} (2\pi i \xi_j) \hat{f}(\xi) e^{-t|2\pi\xi|^{2\alpha}} e^{2\pi i x\cdot\xi} \, dV_n(\xi) = \mathcal{F}^{-1}(\psi_{t,j})(x),$$

where $\psi_{t,j}(\xi) = (2\pi i \xi_j) \hat{f}(\xi) e^{-t|2\pi\xi|^{2\alpha}}$. Therefore, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1} |\nabla_x \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x) \, dt = \int_0^\infty t^{\frac{1}{\alpha}-1} \sum_{j=1}^n \int_{\mathbb{R}^n} |\mathcal{F}^{-1}(\psi_{t,j})(x)|^2 dV_n(x) \, dt$$
$$= \int_0^\infty t^{\frac{1}{\alpha}-1} \sum_{j=1}^n \int_{\mathbb{R}^n} |\psi_{t,j}(\xi)|^2 dV_n(\xi) \, dt = \int_0^\infty t^{\frac{1}{\alpha}-1} \sum_{j=1}^n \int_{\mathbb{R}^n} |2\pi i \xi_j|^2 |\hat{f}(\xi)|^2 e^{-2t|2\pi\xi|^{2\alpha}} dV_n(\xi) \, dt$$

$$= \int_0^\infty t^{\frac{1}{\alpha}-1} \int_{\mathbb{R}^n} |2\pi\xi|^2 |\hat{f}(\xi)|^2 e^{-2t|2\pi\xi|^{2\alpha}} dV_n(\xi)\, dt = 2^{-\frac{1}{\alpha}}\Gamma(\alpha^{-1}) \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 dV_n(\xi).$$

This completes the proof. □

4. Lusin functions of α-parabolic type

We recall the definitions of Lusin functions for $L^{(\alpha)}$-harmonic extensions. For $\xi \in \mathbb{R}^n$ and $\rho > 0$, let

$$C_\rho^{(\alpha)}(\xi) := \{(x,t) \in H : |x-\xi|^{2\alpha} \le \rho^{-1} t\}.$$

Lusin functions for $L^{(\alpha)}$-harmonic extensions are defined by

$$\mathcal{S}_{f,t}^{(\alpha)}(\xi) = \left(\iint_{C_\rho^{(\alpha)}(\xi)} t^{\frac{1}{\alpha}-1-\frac{n}{2\alpha}} |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt \right)^{1/2} \tag{4.1}$$

and

$$\mathcal{S}_{f,x}^{(\alpha)}(\xi) = \left(\iint_{C_\rho^{(\alpha)}(\xi)} t^{\frac{1}{\alpha}-1-\frac{n}{2\alpha}} |\nabla_x \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt \right)^{1/2}. \tag{4.2}$$

We give identities of Lusin type for $L^{(\alpha)}$-harmonic extensions.

THEOREM 4.1. *Let $0 < \alpha \le 1$ and $f \in L^2$. Furthermore, let d_n be the volume of the unit ball of $\mathbb{R}^n$. Then the following identities hold*:

$$\int_{\mathbb{R}^n} |\mathcal{S}_{f,t}^{(\alpha)}(\xi)|^2 dV_n(\xi) = d_n \rho^{-\frac{n}{2\alpha}} 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2 \tag{4.3}$$

$$\int_{\mathbb{R}^n} |\mathcal{S}_{f,x}^{(\alpha)}(\xi)|^2 dV_n(\xi) = d_n \rho^{-\frac{n}{2\alpha}} 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2. \tag{4.4}$$

PROOF. We show the identity (4.3). Let $\Phi_\xi(x,t)$ be the characteristic function of the set $C_\rho^{(\alpha)}(\xi)$. The Fubini theorem implies that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{S}_{f,t}^{(\alpha)}(\xi)|^2 dV_n(\xi) \\
&= \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{\mathbb{R}^n} \Phi_\xi(x,t)\, t^{\frac{1}{\alpha}-1-\frac{n}{2\alpha}} |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt \right) dV_n(\xi) \\
&= \int_0^\infty \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1-\frac{n}{2\alpha}} \left(\int_{\mathbb{R}^n} \Phi_x(\xi,t)\, dV_n(\xi) \right) |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^n} \Phi_x(\xi,t)\, dV_n(\xi) = V_n(C_\rho^{(\alpha)}(x)) = V_n(C_\rho^{(\alpha)}(0)) = d_n \rho^{-\frac{n}{2\alpha}} t^{\frac{n}{2\alpha}},$$

Theorem 3.3 implies that

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{S}_{f,t}^{(\alpha)}(\xi)|^2 dV_n(\xi) &= d_n \rho^{-\frac{n}{2\alpha}} \int_0^\infty \int_{\mathbb{R}^n} t^{\frac{1}{\alpha}-1} |\mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{H}_f^{(\alpha)}(x,t)|^2 dV_n(x)\, dt \\ &= d_n \rho^{-\frac{n}{2\alpha}} 2^{-\frac{1}{\alpha}} \Gamma(\alpha^{-1}) \|f\|_{L^2}^2.\end{aligned}$$

The proof of the identity (4.4) is similar. This completes the proof. □

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