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Boundary behavior of $L^{(\alpha)}$-harmonic extensions of measures

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ABSTRACT. For $0 < \alpha \leq 1$, we consider the parabolic operator $L^{(\alpha)} = \partial/\partial t + (-\Delta_x)^\alpha$ on the upper half-space of the Euclidean space $\mathbb{R}^{n+1}$. In this paper, we study $L^{(\alpha)}$-harmonic extensions of finite signed Borel measures on $\mathbb{R}^n$, which are defined by the convolutions of the fundamental solution of $L^{(\alpha)}$. We obtain a result of classical Fatou type for $L^{(\alpha)}$-harmonic extensions of measures.

1. Introduction

Let $n \geq 1$ and let H be the upper half-space of the $(n+1)$-dimensional Euclidean space, that is, $H = \{X = (x,t) \in \mathbb{R}^{n+1} : x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha, \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $\Delta_x = \partial_1^2 + \cdots + \partial_n^2$. Let $C(H)$ be the set of all real-valued continuous functions on H. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions (for details, see Section 2). In this paper, we study boundary behavior of $L^{(\alpha)}$-harmonic extensions of finite signed Borel measures on $\mathbb{R}^n$, which are defined by the convolutions of the fundamental solution of $L^{(\alpha)}$. The classical Fatou theorem is the following; bounded harmonic functions in the open unit disk have nontangential limits almost everywhere on the unit circle. We obtain a result of classical Fatou type for $L^{(\alpha)}$-harmonic extensions of measures.

To state our main result, we give some definitions. We denote by $W^{(\alpha)}$ the fundamental solution of $L^{(\alpha)}$ (see Section 2 for the definition). Let $M := M(\mathbb{R}^n)$ be the set of all signed Borel measures on $\mathbb{R}^n$ (thus necessarily, $|\mu|(\mathbb{R}^n) < \infty$ for all $\mu \in M$). We define an $L^{(\alpha)}$-harmonic extension $\mathcal{H}_\mu^{(\alpha)}$ of $\mu \in M$ by

$$\mathcal{H}_\mu^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)\,d\mu(y), \qquad (x,t) \in H. \tag{1.2}$$

It is shown that the function $\mathcal{H}_\mu^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H (see [3, Theorem 5.2]). It is well known that when $\alpha = 1/2$, the fundamental solution $W^{(1/2)}$ coincides with the Poisson kernel

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for H (see [4, Section 2]). Therefore, the function $\mathcal{H}_\mu^{(1/2)}$ is the usual harmonic extensions of μ. Let dV_n be the Lebesgue measure on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $\rho > 0$, let

$$C_\rho^{(\alpha)}(x) := \{(y, s) \in H : |y - x|^{2\alpha} \leq \rho^{-1}s\}.$$

The following Fatou theorem is well known([1, Corollary 6.44]).

The Fatou theorem. ([1, Corollary 6.44]) *Let $\mu \in M$. Suppose $d\mu = f dV_n + d\mu_s$ is the Lebesgue decomposition of μ with respect to V_n. If $\rho > 0$, then*

$$\lim_{\substack{(y,s)\to(x,0) \\ (y,s)\in C_\rho^{(1/2)}(x)}} \mathcal{H}_\mu^{(1/2)}(y, s) = f(x)$$

at almost every $x \in \mathbb{R}^n$.

The Fatou theorem shows that the harmonic extension of the measure $d\mu = f dV_n + d\mu_s$ has nontangential limit $f(x)$ at almost every $x \in \mathbb{R}^n$. In this paper, we show the following theorem, which is a result of Fatou type for $\mathcal{H}_\mu^{(\alpha)}$ for all $0 < \alpha \leq 1$.

The main theorem. (Theorem 4.4 of this paper) *Let $0 < \alpha \leq 1$ and $\mu \in M$. Suppose $d\mu = f dV_n + d\mu_s$ is the Lebesgue decomposition of μ with respect to V_n. If $\rho > 0$, then*

$$\lim_{\substack{(y,s)\to(x,0) \\ (y,s)\in C_\rho^{(\alpha)}(x)}} \mathcal{H}_\mu^{(\alpha)}(y, s) = f(x)$$

at almost every $x \in \mathbb{R}^n$.

We describe the construction of this paper. In Section 2, we recall definitions of the $L^{(\alpha)}$-harmonic functions and the fundamental solution of $L^{(\alpha)}$. Furthermore, some lemmas are presented. In Section 3, we present basic properties of the parabolic Hardy spaces. In Section 4, we study the boundary behavior of $\mathcal{H}_\mu^{(\alpha)}$ with $\mu \in M$, and we show a result of Fatou type for $\mathcal{H}_\mu^{(\alpha)}$ for all $0 < \alpha \leq 1$.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Preliminaries

In this section, we recall definitions of the $L^{(\alpha)}$-harmonic functions and the fundamental solution of $L^{(\alpha)}$ (for details, see [4]). We begin with describing the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C^\infty(H) \subset C(H)$ be the set of all infinitely differentiable functions on H and let $C_c^\infty(H) \subset C^\infty(H)$ be the set of all functions in $C^\infty(H)$ with compact support. Then, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(2.1) \qquad (-\Delta_x)^\alpha \psi(x, t) := -C_{n,\alpha} \lim_{\varepsilon\to+0} \int_{|y|>\varepsilon} \frac{\psi(x + y, t) - \psi(x, t)}{|y|^{n+2\alpha}} dV_n(y)$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma\big((n+2\alpha)/2\big)/\Gamma(-\alpha) > 0$ and Γ is the gamma function. Let $\widetilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is,

$$\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)\widetilde{L}^{(\alpha)}\psi(x,t)| dV_n(x)dt < \infty \quad \text{and} \quad \int_0^\infty \int_{\mathbb{R}^n} u(x,t)\widetilde{L}^{(\alpha)}\psi(x,t) dV_n(x)dt = 0$$

for all $\psi \in C_c^\infty(H)$.

We present the explicit definition of the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x\cdot\xi)\, dV_n(\xi) & (t > 0) \\ 0 & (t \le 0), \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi\cdot\xi)^{1/2}$. The function $W^{(\alpha)}$ is called the fundamental solution of $L^{(\alpha)}$. We also describe basic properties of $W^{(\alpha)}$. It is well known that

$$W^{(\alpha)}(x,t) > 0, \qquad (x,t) \in H \tag{2.2}$$

and

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x,t) dV_n(x) = 1, \qquad 0 < t < \infty. \tag{2.3}$$

We also remark that $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H and $W^{(\alpha)} \in C^\infty(H)$. The following estimate is [4, Lemma 3.1]: there exists a constant $C = C(n,\alpha) > 0$ such that

$$W^{(\alpha)}(x,t) \le C \frac{t}{(t + |x|^{2\alpha})^{n/2\alpha+1}} \tag{2.4}$$

for all $(x,t) \in H$.

In case $\alpha = 1/2$, the function $W^{(1/2)}$ is the Poisson kernel, that is,

$$W^{(1/2)}(x,t) = \begin{cases} \dfrac{\Gamma\big((n+1)/2\big)}{\pi^{(n+1)/2}} \dfrac{t}{(t^2 + |x|^2)^{(n+1)/2}} & (t > 0) \\ 0 & (t \le 0). \end{cases}$$

In case $\alpha = 1$, the function $W^{(1)}$ is the Gauss kernel, that is,

$$W^{(1)}(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\Big(-\dfrac{|x|^2}{4t}\Big) & (t > 0) \\ 0 & (t \le 0). \end{cases}$$

In other cases, simple explicit expressions for $W^{(\alpha)}$ are not known.

3. The parabolic Hardy spaces

The parabolic Hardy spaces were introduced in [4, Remark 5.7]. Some properties of their spaces have been studied in [2]. In this section, we present basic properties of the parabolic Hardy spaces. We begin with recalling definition of the parabolic Hardy spaces. For $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$, the α-parabolic Hardy space $\boldsymbol{h}^p_\alpha$ is the set of all $L^{(\alpha)}$-harmonic functions u on H with

$$\|u\|_{\boldsymbol{h}^p_\alpha} := \sup_{t>0} \|u(\,\cdot\,, t)\|_{L^p} = \sup_{t>0} \left(\int_{\mathbb{R}^n} |u(x,t)|^p dV_n(x) \right)^{1/p} < \infty.$$

We remark that by Lemma 2.2 of [2] (see Section 4 of [2])

$$\|u\|_{\boldsymbol{h}^p_\alpha} = \lim_{t\to 0} \|u(\,\cdot\,, t)\|_{L^p}$$

for all $1 \leq p \leq \infty$ and $u \in \boldsymbol{h}^p_\alpha$.

For $1 \leq p \leq \infty$, the Lebesgue space $L^p := L^p(\mathbb{R}^n, dV_n)$ is defined to be the Banach space of Lebesgue measurable (real-valued) functions on $\mathbb{R}^n$ with norm $\|\cdot\|_{L^p}$. If $d\mu = f dV_n$ with $f \in L^p$, we will write $\mathcal{H}^{(\alpha)}_\mu = \mathcal{H}^{(\alpha)}_f$, and we say that $\mathcal{H}^{(\alpha)}_f$ is the $L^{(\alpha)}$-harmonic extension of the funtion f. We present the following lemma, which is a result of [2].

LEMMA 3.1. ([2, Theorem 4.1]) *Let* $0 < \alpha \leq 1$. *Then, the following statements hold*:

(1) *For* $1 < p \leq \infty$, *the mapping* $f \mapsto \mathcal{H}^{(\alpha)}_f$ *is a linear isometry of* L^p *onto* $\boldsymbol{h}^p_\alpha$.

(2) *The mapping* $\mu \mapsto \mathcal{H}^{(\alpha)}_\mu$ *is a linear isometry of* M *onto* $\boldsymbol{h}^1_\alpha$.

We also describe the definition of the α-parabolic maximal function, which is introduced in [2]. We recall the definition of $C^{(\alpha)}_\rho(x)$. For $x \in \mathbb{R}^n$ and $\rho > 0$, let

$$C^{(\alpha)}_\rho(x) := \{(y,s) \in H : |y-x|^{2\alpha} \leq \rho^{-1}s\}.$$

The α-parabolic maximal function $\mathcal{N}^{(\alpha)}_\rho[u]$ of a function u on H is defined by

$$\mathcal{N}^{(\alpha)}_\rho[u](x) := \sup\bigl\{|u(y,s)| : (y,s) \in C^{(\alpha)}_\rho(x)\bigr\}, \qquad x \in \mathbb{R}^n.$$

Clearly, for a function u on H, we have

$$\|u\|_{\boldsymbol{h}^p_\alpha} \leq \|\mathcal{N}^{(\alpha)}_\rho[u]\|_{L^p} \tag{3.1}$$

for all $0 < \alpha \leq 1$, $1 \leq p \leq \infty$, and $\rho > 0$. The following result is Theorem 7.3 of [2].

LEMMA 3.2. ([2, Theorem 7.3]) *Let* $0 < \alpha \leq 1$ *and* $\rho > 0$. *Then, the following statements hold*:

(1) *If* $1 < p \leq \infty$, *then there exists a constant* $C = C(n, \alpha, p, \rho) > 0$ *such that*

$$\|u\|_{\boldsymbol{h}^p_\alpha} \leq \|\mathcal{N}^{(\alpha)}_\rho[u]\|_{L^p} \leq C\|u\|_{\boldsymbol{h}^p_\alpha}$$

for all $L^{(\alpha)}$*-harmonic functions* u *on* H.

(2) *If* $p = 1$*, then there exists a constant* $C = C(n, \alpha, \rho) > 0$ *such that*

$$V_n(\{x \in \mathbb{R}^n : \mathcal{N}_\rho^{(\alpha)}[u](x) > \lambda\}) \leq \frac{C}{\lambda} \|u\|_{\boldsymbol{h}_\alpha^1}$$

for all $\lambda > 0$ *and* $L^{(\alpha)}$*-harmonic functions* u *on* H.

4. The boundary behavior of the $L^{(\alpha)}$-harmonic extensions

In this section, we study the boundary behavior of $\mathcal{H}_\mu^{(\alpha)}$ with $\mu \in M$, and we show a result of Fatou type for $\mathcal{H}_\mu^{(\alpha)}$ for all $0 < \alpha \leq 1$. First, we describe a result concerning the boundary behavior of $\mathcal{H}_f^{(\alpha)}$ with $f \in L^p$, which is obtained in [5]. For $1 \leq p \leq \infty$ and $f \in L^p$, let $\mathcal{S}(f)$ be the set of all points $x \in \mathbb{R}^n$ which satisfy the following condition; for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_{B(y,r)} \big(f(z) - f(x)\big) dV_n(z) \right| < \varepsilon(|y - x| + r)^n$$

for all $y \in \mathbb{R}^n$ with $|y - x| < \delta$ and $r < \delta$, where $B(y, r)$ is the Euclidean ball of radius r and center at y. We give the following remark.

REMARK 4.1. For $1 \leq p \leq \infty$ and $f \in L^p$, every Lebesgue point of f belongs to $\mathcal{S}(f)$. In fact, suppose that $x \in \mathbb{R}^n$ is a Lebesgue point of f, and let $\varepsilon > 0$. Then, there exists $\delta > 0$ such that

$$\int_{B(x,\eta)} \big|f(z) - f(x)\big| dV_n(z) < \varepsilon\eta^n$$

for all $\eta < 2\delta$. Thus, if $|y - x| < \delta$ and $r < \delta$, then $\eta := |y - x| + r < 2\delta$, so we have

$$\begin{aligned} \left| \int_{B(y,r)} \big(f(z) - f(x)\big) dV_n(z) \right| &\leq \int_{B(y,r)} \big|f(z) - f(x)\big| dV_n(z) \\ &\leq \int_{B(x,|y-x|+r)} \big|f(z) - f(x)\big| dV_n(z) < \varepsilon(|y - x| + r)^n. \end{aligned}$$

Therefore, x belongs to $\mathcal{S}(f)$.

We describe the main theorem in [5], which is concerned with the boundary behavior of $\mathcal{H}_f^{(\alpha)}$ with $f \in L^p$ (actually the result in [5] is more general).

LEMMA 4.2. ([5, The main theorem]) *Let* $0 < \alpha \leq 1$*,* $1 \leq p \leq \infty$*, and* $f \in L^p$*. If* $\rho > 0$*, then*

$$\lim_{\substack{(y,s)\to(x,0) \\ (y,s)\in C_\rho^{(\alpha)}(x)}} \mathcal{H}_f^{(\alpha)}(y, s) = f(x)$$

for all $x \in \mathcal{S}(f)$.

By Remark 4.1 and Lemma 4.2, we have the following corollary, which is a result of Fatou type.

COROLLARY 4.3. *Let* $0 < \alpha \leq 1$, $1 \leq p \leq \infty$, *and* $f \in L^p$. *If* $\rho > 0$, *then*

$$\lim_{\substack{(y,s)\to(x,0)\\(y,s)\in C_\rho^{(\alpha)}(x)}} \mathcal{H}_f^{(\alpha)}(y,s) = f(x)$$

at almost every $x \in \mathbb{R}^n$.

For $\mu \in M$, we put

$$L_\mu(x) := \limsup_{\substack{(y,s)\to(x,0)\\(y,s)\in C_\rho^{(\alpha)}(x)}} \mathcal{H}_\mu^{(\alpha)}(y,s), \qquad x \in \mathbb{R}^n.$$

We show the following theorem, which is a result of Fatou type for $\mathcal{H}_\mu^{(\alpha)}$ with $\mu \in M$.

THEOREM 4.4. *Let* $0 < \alpha \leq 1$ *and* $\mu \in M$. *Suppose* $d\mu = f dV_n + d\mu_s$ *is the Lebesgue decomposition of* μ *with respect to* V_n. *If* $\rho > 0$, *then*

$$\lim_{\substack{(y,s)\to(x,0)\\(y,s)\in C_\rho^{(\alpha)}(x)}} \mathcal{H}_\mu^{(\alpha)}(y,s) = f(x)$$

at almost every $x \in \mathbb{R}^n$.

PROOF. By Corollary 4.3 and the Jordan decomposition, it suffices to show that if $\mu \in M$ is positive and $\mu \perp V_n$ then $L_\mu(x) = 0$ at almost every $x \in \mathbb{R}^n$. In particular, we shall show that $V_n(\{L_\mu > 2\lambda\}) = 0$ for each $\lambda > 0$. Thus, suppose $\mu \in M$ is positive, $\mu \perp V_n$, and $\lambda > 0$. Let $\varepsilon > 0$. Then, since μ is regular, there exists an open set $E \subset \mathbb{R}^n$ such that $\mu(E) < \varepsilon$ and $V_n(\mathbb{R}^n \backslash E) = 0$. Put $d\omega := \mathcal{X}_E d\mu$ and $d\sigma := \mathcal{X}_{\mathbb{R}^n\backslash E} d\mu$. Since

$$\{L_\mu > 2\lambda\} \subset \{L_\omega > \lambda\} \cup \{L_\sigma > \lambda\},$$

we shall give estimates of $V_n(\{L_\omega > \lambda\})$ and $V_n(\{L_\sigma > \lambda\})$.

First, we shall estimate $V_n(\{L_\omega > \lambda\})$. Put $u = \mathcal{H}_\omega^{(\alpha)}$. Then, by the definition, we have $L_\omega(x) \leq \mathcal{N}_\rho^{(\alpha)}[u](x)$ for all $x \in \mathbb{R}^n$. Thus, Lemma 3.2 (2) and Theorem 3.1 (2) imply

$$V_n(\{L_\omega > \lambda\}) \leq V_n(\{\mathcal{N}_\rho^{(\alpha)}[u] > \lambda\}) \leq C\lambda^{-1}\|u\|_{\boldsymbol{h}_\alpha^1} = C\lambda^{-1}\|\omega\|.$$

Since $\mu(E) < \varepsilon$ and $d\omega = \mathcal{X}_E d\mu$, we obtain $V_n(\{L_\omega > \lambda\}) \leq C\lambda^{-1}\varepsilon$.

Next, we show $V_n(\{L_\sigma > \lambda\}) = 0$. First, we claim that

$$\lim_{(y,s)\to(x,0)} \mathcal{H}_\sigma^{(\alpha)}(y,s) = 0 \tag{4.1}$$

for all $x \in E$. In fact, let $x \in E$ be fixed. Since E is an open set, there exists $\delta > 0$ such that $B(x, 2\delta) \subset E$. It is enough to consider the behavior of $\mathcal{H}_\sigma^{(\alpha)}(y,s)$ with (y,s) sufficiently near $(x,0)$. Therefore, suppose $|x-y| < \delta$. Then, by (2.2) and (2.4), we have

$$\begin{aligned}\mathcal{H}_\sigma^{(\alpha)}(y,s) &\leq \int_{\mathbb{R}^n \setminus B(x,2\delta)} W^{(\alpha)}(y-z,s)d\mu(z) \leq C \int_{\mathbb{R}^n \setminus B(x,2\delta)} \frac{s}{(s+|y-z|^{2\alpha})^{n/2\alpha+1}} d\mu(z) \\ &\leq C \int_{\mathbb{R}^n \setminus B(x,2\delta)} \frac{s}{(|y-z|^{2\alpha})^{n/2\alpha+1}} d\mu(z) \leq C \frac{s}{\delta^{n+2\alpha}} \|\mu\|,\end{aligned}$$

because $|y-z| \geq |z-x| - |x-y| > 2\delta - \delta = \delta$ for all $z \in \mathbb{R}^n \setminus B(x, 2\delta)$. Hence, we obtain (4.1) for all $x \in E$. Since this implies $\{L_\sigma > \lambda\} \subset \mathbb{R}^n \setminus E$, we conclude $V_n(\{L_\sigma > \lambda\}) \leq V_n(\mathbb{R}^n \setminus E) = 0$.

By the above estimates, we have $V_n(\{L_\mu > 2\lambda\}) \leq C\lambda^{-1}\varepsilon$. Since ε is arbitrary, we obtain $V_n(\{L_\mu > 2\lambda\}) = 0$. This completes the proof. □

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