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THE REGION INDEX AND THE RASMUSSEN INVARIANT OF A KNOT

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ABSTRACT. For a knot in the 3-sphere S^3, there exists a diagram such that we have the unknot by changing all crossings on the boundary of some region of the diagram. The minimal number of the crossing changes over all such diagrams is called the region index of a knot. In this paper, we show that there exists infinite families of knots which have the region indices equal to two and three by using the Rasmussen invariant.

Key words: Knots; unknotting number; Rasmussen invariant.

1. Introduction

Let D be a diagram of a knot in S^3 and let D^+ be the four-valent graph which is obtained from D by replacing each crossing with a vertex. We call the closure of each connected component of $S^2 - D^+$ a *region* of D. A *region crossing change* at a region R of D is the local deformation on D by the changing all the crossings on the boundary of R [8]. For any knot K, it is known that there exists a diagram D such that D can be transformed into a diagram of the unknot by performing a single region crossing change to some region R of D [1]. The *region index* of a knot K, denoted by $\mathrm{Reg}(K)$, is defined to be the minimal number of crossings on the boundary of an unknotting region of an unknotting region diagram of K over all such diagrams. The region index has been introduced by A. Kawauchi, K. Kishimoto and A. Shimizu. The *unknotting number* of a knot K, denoted by $u(K)$, is the minimal number of crossing changes needed to create the unknot, the minimum being taken over all possible sets of changes in all diagrams of K [4][5]. The unknotting number is less than or equal to the region index. In particular, we can easily see that the region index of any knot with unknotting number one is larger then or equal to two by the definition. In general, it is not easy to calculate the region index. In this paper, we obtain the following results.

Theorem 1.1. *There exists an infinite family of knots* $\{K_m\}$ *in* S^3 *such that* $\mathrm{Reg}(K_m) = 2$.

Theorem 1.2. *There exists an infinite family of knots* $\{K_n\}$ *in* S^3 *such that* $\mathrm{Reg}(K_n) = 3$.

In Sections 2, we shall give proofs of Theorems 1.1 and 1.2. In Section 3, we shall show examples of calculation of the region index.

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2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let m be a positive integer. We consider the knot K_m which has a diagram as in Figure 1. We can calculate that the Goeritz invariant of K_m is equal to $(4m + 1)$ [2][3]. In particular, it is non-trivial. Thus $2 \leq \mathrm{Reg}(K_m)$ by the definition of the region index. On the other hand, if we perform a region crossing change at the region R as indicated in Figure 1, then we have a diagram of the unknot. So we know that $\mathrm{Reg}(K_m) \leq 2$. Thus we have $\mathrm{Reg}(K_m) = 2$. This completes the proof.

Proof of Theorem 1.2. Let n be a positive odd integer. We consider the knot K_n which has a diagram as in Figure 2. Then we know that K_n is a positive knot and the Rasmussen invariant is equal to three by Theorem 4 in [7]. Thus by Theorem 1 in [7], we know that $3 \leq g_*(K_n) \leq u(K_n)$ where

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$g_*(K_n)$ is the smooth slice genus of K_n. On the other hand, if we perform a region crossing change at the region R as indicated in Figure 2, then we have a diagram of the unknot. So we know that $\mathrm{Reg}(K_n) \leq 3$. Thus we have $\mathrm{Reg}(K_n) = 3$. We can also calculate that the Goeritz invariant of K_n is equal to $(4n-1)$ [2][3]. This completes the proof.

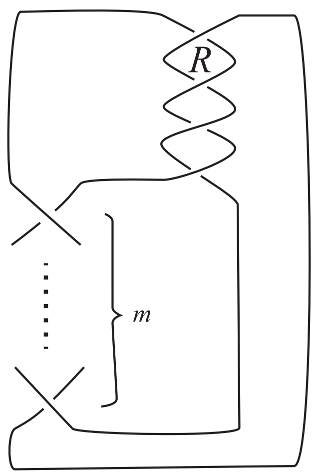

FIGURE 1. The knot K_m

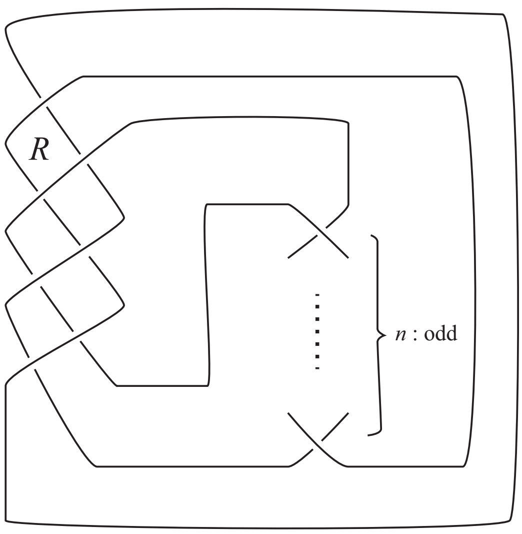

FIGURE 2. The knot K_n

3. EXAMPLES

Example 3.1. $\mathrm{Reg}(8_{19}) = 3$.

The knot 8_{19} is the knot which has a diagram as in Figure 3. If we perform a region crossing change at R as indicated in Figure 3, then we have the unknot. So we know that $\mathrm{Reg}(8_{19}) \leq 3$. On the other hand, the unknotting number is known to be three [4]. Thus we know that $\mathrm{Reg}(8_{19}) = 3$.

Example 3.2. $\mathrm{Reg}(9_{38}) = 3$.

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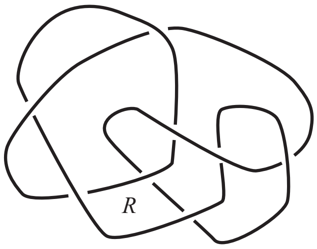

FIGURE 3. The knot 8_{19}

The knot 9_{38} is the knot which has a diagram as in Figure 4. By performing a region crossing change at R, then we have the unknot. So we know that $\mathrm{Reg}(9_{38}) \leq 3$. On the other hand, Owens showed that $u(9_{38}) = 3$ [6]. Thus we know that $\mathrm{Reg}(9_{38}) = 3$.

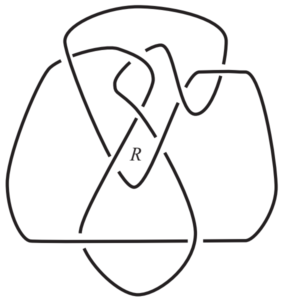

FIGURE 4. The knot 9_{38}

Example 3.3. $\mathrm{Reg}(9_{49}) = 3$.

The knot 9_{49} is the knot as in Figure 5. If we perform a region crossing change at R, then we have the unknot. So we know that $\mathrm{Reg}(9_{49}) \leq 3$. On the other hand, Stoimenow showed that $u(9_{49}) = 3$ [9]. Thus we know that $\mathrm{Reg}(9_{49}) = 3$.

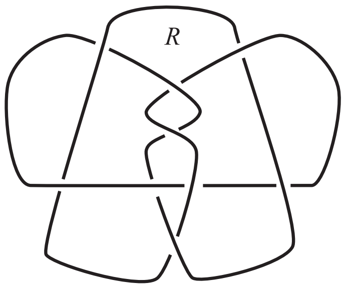

FIGURE 5. The knot 9_{49}

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