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# Parabolic Cohomology Groups and Periods of Cusp Forms

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**Abstract.** Main results of this paper are Theorems 9, 10, 11 and Proposition 12 below. Let $\Gamma = \Gamma_1(N)$ and $m = (SL(2,\mathbb{Z}):\Gamma)$. Let $S_{w+2}(\Gamma)$ =the space of cusp forms of integral weight $w+2 \geq 2$ on $\Gamma$. We study the parabolic cohomology groups of real and integral coefficients associated with $S_{w+2}(\Gamma)$. We represent them on the Euclidean space $\mathbb{R}^{(w+1)m}$ containing vectors

$$\left( \mathrm{Re} \int_0^{i\infty} f\,|_{w+2}\,[g](z) z^k dz \right)_{k\in[0,w] \text{ and } \Gamma g \in \Gamma \backslash SL(2,\mathbb{Z})}$$

of all $f \in S_{w+2}(\Gamma)$.

## Introduction and Notations

Let $N$ be any positive integer and let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, a-1 \equiv d-1 \equiv c \equiv 0 \pmod N \right\} .$$

This is Hecke's congruence subgroup of $SL(2,\mathbb{Z})$. Write $\mathbf{t} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$\Gamma = \Gamma_1(N)$ , $\Gamma' = \Gamma_1(N)\{\pm \mathbf{1}_2\}$ and $\Gamma'' = \Gamma_1(N)\{\pm\mathbf{1}_2, \pm\mathbf{t}\} \subset GL(2,\mathbb{Z})$ for short.

($\{\pm\mathbf{1}_2,\pm\mathbf{t}\}$ is a subgroup of $GL(2,\mathbb{Z})$, and $\Gamma_1(N)\{\pm\mathbf{1}_2,\pm\mathbf{t}\}=\{\pm\mathbf{1}_2,\pm\mathbf{t}\}\Gamma_1(N)$ that is a subgroup of $GL(2,\mathbb{Z})$.) Let $\chi_0:\Gamma''\to\{1\}\subset\mathbb{C}^*$ denote the trivial character of $\Gamma''$. If $N\geq 3$, let $\chi_1:\Gamma''\to\{\pm 1\}\subset\mathbb{C}^*$ denote the character given by $\chi_1(\gamma)=1$, $\chi_1(-\gamma)=-1$, $\chi_1(\gamma\mathbf{t})=-1$, $\chi_1(-\gamma\mathbf{t})=1$ for any $\gamma\in\Gamma$. If $N\leq 2$, let $\chi_1=\chi_0$. Let $w$ denote any integer $\geq 2$. Let $\rho_w$ : $\mathrm{GL}(2,\mathbb{Z})\to\mathrm{GL}(w+1,\mathbb{Z})$ denote the polynomial representation

$$\text{given by } (ad-bc)\begin{pmatrix}(cX+d)^w\\(aX+b)^1(cX+d)^{w-1}\\ \vdots \\(aX+b)^{w-1}(cX+d)\\(aX+b)^w\end{pmatrix}=\rho_w\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)\begin{pmatrix}1\\X\\ \vdots\\X^{w-1}\\X^w\end{pmatrix}.$$

Let $S_{w+2}(\Gamma)$ denote the space of cusp forms of weight $w+2$ with respect to $\Gamma$. Let

$$S_{w+2}(\Gamma',\chi|_{\Gamma'})=\left\{f\in S_{w+2}(\Gamma)\;\middle|\; f\left(\frac{az+b}{cz+d}\right)(cz+d)^{w+2}=\chi(\gamma)f(z)\text{ for all }\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma'\right\}$$

where $\chi\in\{\chi_0,\chi_1\}$. By the representation theory of Abelian groups, we have

$$S_{w+2}(\Gamma)=S_{w+2}(\Gamma',\chi_0|_{\Gamma'})\oplus_{\mathbb{C}}S_{w+2}(\Gamma',\chi_1|_{\Gamma'})\quad\text{if } N\geq 3.$$

We have $S_{w+2}(\Gamma',\chi_0|_{\Gamma'})=\{0\}$ if $w$ is odd and $N\geq 3$;
$S_{w+2}(\Gamma',\chi_1|_{\Gamma'})=\{0\}$ if $w$ is even and $N\geq 3$.

We put

$S_{w+2}^{\mathbb{R}}(\Gamma)=\{f(z)\in S_{w+2}(\Gamma)\mid$ Fourier coefficients of $f(z)$ at $z=i\infty$ are all real numbers$\}$ and

$S_{w+2}^{\mathbb{R}}(\Gamma',\chi|_{\Gamma'})=\{f(z)\in S_{w+2}(\Gamma',\chi|_{\Gamma'})\mid$ Fourier coefficients of $f(z)$ at $z=i\infty$ are all real numbers$\}$

where $\chi\in\{\chi_0,\chi_1\}$. Then we have also

$$S_{w+2}^{\mathbb{R}}(\Gamma)=S_{w+2}^{\mathbb{R}}(\Gamma',\chi_0|_{\Gamma'})\oplus_{\mathbb{C}}S_{w+2}^{\mathbb{R}}(\Gamma',\chi_1|_{\Gamma'})\quad\text{if } N\geq 3.$$

If $N\leq 2$, we have $\Gamma=\Gamma'$ and $S_{w+2}(\Gamma)=S_{w+2}(\Gamma')=S_{w+2}(\Gamma',\chi)$.

If $N \le 2$, we have $\Gamma = \Gamma'$ and $S_{w+2}(\Gamma) = S_{w+2}(\Gamma') = S_{w+2}(\Gamma', \chi)$.

If $N \le 2$ and $w$ is odd, we have $S_{w+2}(\Gamma) = \{0\}$.

If $w$ is even, we consider $\rho_w|_{\Gamma'}$ (resp. $\rho_w|_{\Gamma''}$) and $\mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}(\rho_w|_{\Gamma'})$ (resp. $\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}(\rho_w|_{\Gamma''})$). If $w$ is odd, we consider $(\chi_1 \otimes \rho_w)|_{\Gamma'}$ (resp. $(\chi_1 \otimes \rho_w)|_{\Gamma''}$) and $\mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}((\chi_1 \otimes \rho_w)|_{\Gamma'})$ (resp. $\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}((\chi_1 \otimes \rho_w)|_{\Gamma''})$). Note that a system of representatives for $\Gamma' \backslash SL(2,\mathbb{Z})$ are a system of representatives for $\Gamma'' \backslash GL(2,\mathbb{Z})$. Then we have $(\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}(\rho_w|_{\Gamma''}))|_{SL(2,\mathbb{Z})} = \mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}(\rho_w|_{\Gamma'})$ if $w$ is even;

$(\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}((\chi_1 \otimes \rho_w)|_{\Gamma''}))|_{SL(2,\mathbb{Z})} = \mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}((\chi_1 \otimes \rho_w)|_{\Gamma'})$ if $w$ is odd.

Let $\{g_j\}_{j=1}^m$ be a system of representatives for $\Gamma' \backslash SL(2,\mathbb{Z})$ with $g_1 = \mathbf{1}_2$. Put

$$d\mathbf{z}_w = \begin{pmatrix} dz \\ zdz \\ z^2 dz \\ \vdots \\ z^w dz \end{pmatrix}$$ which is a differential form on the complex upper half plane.

We normalize $\mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}(\rho_w|_{\Gamma'})$ (resp. $\mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}((\chi_1 \otimes \rho_w)|_{\Gamma'})$) as $\eta_w$ in the following way if $S_{w+2}(\Gamma) \ne \{0\}$. Let $f(z)$ denote any cusp form in $S_{w+2}(\Gamma)$.

$$\eta_w\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)\begin{pmatrix} (f(z)d\mathbf{z}_w)\circ g_1 \\ (f(z)d\mathbf{z}_w)\circ g_2 \\ \vdots \\ (f(z)d\mathbf{z}_w)\circ g_{m-1} \\ (f(z)d\mathbf{z}_w)\circ g_m \end{pmatrix} = \begin{pmatrix} (f(z)d\mathbf{z}_w)\circ g_1 \\ (f(z)d\mathbf{z}_w)\circ g_2 \\ \vdots \\ (f(z)d\mathbf{z}_w)\circ g_{m-1} \\ (f(z)d\mathbf{z}_w)\circ g_m \end{pmatrix}\circ\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma'$.

Then we have $\eta_w(-\mathbf{1}_2)=1$. Since $(\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}(\rho_w|_{\Gamma''}))|_{SL(2,\mathbb{Z})}=\mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}(\rho_w|_{\Gamma'})$

(resp. $(\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}((\chi_1\otimes\rho_w)|_{\Gamma''}))|_{SL(2,\mathbb{Z})}=\mathrm{Ind}_{\Gamma'}^{SL(2,\mathbb{Z})}((\chi_1\otimes\rho_w)|_{\Gamma'})$) if $w$ is even (resp.

odd), we can extend the domain of $\eta_w$ to $GL(2,\mathbb{Z})$ uniquely as $\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}(\rho_w|_{\Gamma''})$

(resp. $\mathrm{Ind}_{\Gamma''}^{GL(2,\mathbb{Z})}((\chi_1\otimes\rho_w)|_{\Gamma''}))$) if $w$ is even (resp. odd). Write

$$D(f)=\begin{pmatrix}(f(z)d\mathbf{z}_w)\circ g_1\\(f(z)d\mathbf{z}_w)\circ g_2\\\vdots\\(f(z)d\mathbf{z}_w)\circ g_{m-1}\\(f(z)d\mathbf{z}_w)\circ g_m\end{pmatrix},\ D(f)^{\varepsilon}=\begin{pmatrix}(\rho_w(\mathbf{t}))(f(z)d\mathbf{z}_w)\circ\mathbf{t}g_1\mathbf{t}\\(\rho_w(\mathbf{t}))(f(z)d\mathbf{z}_w)\circ\mathbf{t}g_2\mathbf{t}\\\vdots\\(\rho_w(\mathbf{t}))(f(z)d\mathbf{z}_w)\circ\mathbf{t}g_{m-1}\mathbf{t}\\(\rho_w(\mathbf{t}))(f(z)d\mathbf{z}_w)\circ\mathbf{t}g_m\mathbf{t}\end{pmatrix},\ s_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$

$s_2=\begin{pmatrix}1&-1\\1&0\end{pmatrix}$. Let $P$ denote the set consisting of all the parabolic elements in

$SL(2,\mathbb{Z})$. Let $H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},R)$, $H^1_P(SL(2,\mathbb{Z}),\eta_w,R)$, $Z^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},R)$,

$Z^1_P(SL(2,\mathbb{Z}),\eta_w,R)$ denote the parabolic cohomology groups and the groups of parabolic cocycles with $R$-coefficients respectively where $R\in\{\mathbb{R},\mathbb{Z}\}$. The natural injection $\mathbb{Z}\to\mathbb{R}$ induces the canonical homomorphisms

$j_1: H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})\to H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ and

$j_2: H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{Z})\to H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{R})$ where $\chi\in\{\chi_0,\chi_1\}$. The image

of $j_1$ (resp. $j_2$) is a lattice of $H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ (resp. $H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{R})$). Let $\varphi$ denote the Eichler-Shimura $\mathbb{R}$-linear isomorphism :

$S_{w+2}(\Gamma)=S_{w+2}(\Gamma',\chi)\to H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{R})$ where $\chi=\chi_0$ if $2|w$, and $\chi=\chi_1$ if $w$ is odd. (Cf. Shimura [11]). Put

$H(\Gamma,\mathbb{Z})^+=\varphi^{-1}(j_2(H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{Z})))\cap S^{\mathbb{R}}_{w+2}(\Gamma)$ and

$$H(\Gamma,\mathbb{Z})^- = \varphi^{-1}(j_2(H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{Z})))\cap\sqrt{-1}S^{\mathbb{R}}_{w+2}(\Gamma)$$

where $\chi=\chi_0$ if $2\,|\,w$, and $\chi=\chi_1$ if $w$ is odd. Let $\varepsilon$ be the map :

$S_{w+2}(\Gamma)\to S_{w+2}(\Gamma)$ given by $\varepsilon(f(z))=\overline{f(-\bar{z})}$. Namely

$$\varepsilon\left(\sum_{n=1}^{\infty}a_n\exp(2\pi inz)\right)=\sum_{n=1}^{\infty}\overline{a_n}\exp(2\pi inz).$$

## 1. Our results and proofs

We have

**Proposition 1.** *The group* $\varphi^{-1}(j_2(H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{Z})))$ *is stable by the map* $\varepsilon$ *where* $\chi=\chi_0$ *if* $2\,|\,w$, *and* $\chi=\chi_1$ *if* $w$ *is odd.*

We may omit the proof of Lemma 1 since it goes in the same way as the proof of [6, Lemma 1.6, p.16] by replacing $\Gamma$ (resp. $\rho_w|_{\Gamma}$) by $\Gamma'$ (resp. $\chi\otimes\rho_w|_{\Gamma'}$).

**Lemma 2.** *Let* $f(z)\in S_{w+2}(\Gamma)$. *Let* $z_0$ *be either a point in the complex upper half plane or a cusp of* $\Gamma$. *The map* $SL(2,\mathbb{Z})\to\mathbb{R}^{(w+1)m}$, $g\mapsto \mathrm{Re}\int_{z_0}^{g(z_0)}D(f)$ *is a parabolic cocycle in* $Z^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$. *Its cohomology class* $\Phi(f)$ *in* $H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ *is determined by* $f$ *and independent of choice of* $z_0$.

The proof of Lemma 2 is the same as the proof of [6, Lemma 2.1, pp. 18-19].

By Lemma 2 we have the $\mathbb{R}$-linear map $\Phi: S_{w+2}(\Gamma)\to H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$.

Let $R$ be a commutative ring and let $V_R$ be the $R$-module consisting of all the $R^{w+1}$ valued functions $h$ on $SL(2,\mathbb{Z})$ such that $h(\mathbf{xy})=(\chi\otimes\rho_w)(\mathbf{x})h(\mathbf{y})$ for all

$\mathbf{x} \in \Gamma'$ and all $\mathbf{y} \in SL(2,\mathbb{Z})$, where $\chi = \chi_0$ if $2\,|\,w$, and $\chi = \chi_1$ if $w$ is odd. Let $\{g_j\}_{j=1}^{m}$ be a system of representatives for $\Gamma' \backslash SL(2,\mathbb{Z})$ with $g_1 = \mathbf{1}_2$. Then we have :

**Lemma 3** (Shapiro's lemma). For each non-negative integer $n$, the map induced by the compatible maps $\Gamma' \subset SL(2,\mathbb{Z})$ and

$$\mathrm{sh}^n R : H^n(SL(2,\mathbb{Z}),\eta_w,R) \to H^n(\Gamma',\chi \otimes \rho_w|_{\Gamma'},R), \qquad h\,(\in V_R) \mapsto h(g_1) \in R^{w+1}$$

is a surjective isomorphism as $R$-modules.

Let $\mathrm{sh}_P R$ denote $\mathrm{sh}^1 R|_{H_P^1(SL(2,Z),\eta_w,R)}$.

**Theorem 4.1.** *Let* $R$ *denote a commutative ring. Let* $\mathbb{R}$ *denote the field of all the real numbers.*

(i) *(The image of* $\mathrm{sh}_P R$*)* $\subset H^1_{P\cap\Gamma'}(\Gamma',\chi \otimes \rho_w|_{\Gamma'},R)$.

(ii) *The composite map* $\mathrm{sh}_P\mathbb{R} \circ \Phi$ *is equal to the Eichler-Shimura* $\mathbb{R}$*-linear isomorphism* $\varphi$ *above.*

(iii) $\mathrm{sh}_P\mathbb{R} : H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R}) \to H^1_{P\cap\Gamma'}(\Gamma',\chi \otimes \rho_w|_{\Gamma'},\mathbb{R})$ *is a surjective* $\mathbb{R}$*-linear isomorphism.*

(iv) *The* $\mathbb{R}$*-linear map* $\Phi : S_{w+2}(\Gamma) \to H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ *is bijective.*

*Here* $\chi = \chi_0$ *if* $2\,|\,w$, *and* $\chi = \chi_1$ *if* $w$ *is odd.*

The proof of Theorem 4.1 goes in the same way as those of [6, Theorems 2.2 and 2.3]. We may leave it to the reader.

**Theorem 5.1.** *The map* $\mathrm{sh}_P\mathbb{Z} : H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z}) \to H^1_{P\cap\Gamma'}(\Gamma',\chi \otimes \rho_w|_{\Gamma'},\mathbb{Z})$ *is a surjective* $\mathbb{Z}$*-linear isomorphism. Here* $\chi = \chi_0$ *if* $2\,|\,w$, *and* $\chi = \chi_1$ *if* $w$ *is odd.*

The proof of Theorem 5.1 goes in the same way as that of [6, Lemma 2.7]. We may leave it to the reader.

**Corollary of Theorem 5.1 and Theorem 4.1 (iii).** *The restriction of the domain of*

$\mathrm{sh}_P\mathbb{R}: H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R}) \to H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{R})$ *to* $j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z}))$

*is a surjective isomorphism from* $j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z}))$ *to* $j_2(H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{Z}))$.

*Here* $\chi=\chi_0$ *if* $2\,|\,w$, *and* $\chi=\chi_1$ *if* $w$ *is odd.*

It is well known that the group $SL(2,\mathbb{Z})$ is generated by $s_1$ and $s_2$ with the relations $s_1^2=s_2^3=-\mathbf{1}_2$. Any cocycle $C(g)$ $(g\in SL(2,\mathbb{Z}))$ in $Z^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ is determined by $C(s_1)$ and $C(s_2)$. We have $C(\mathbf{1}_2)=0$ since $\mathbf{1}_2^2=\mathbf{1}_2$. We have also $C(-\mathbf{1}_2)=0$ since $(-\mathbf{1}_2)^2=\mathbf{1}_2$ and $\eta_w(-\mathbf{1}_2)=\mathbf{1}$. Apply Lemma 2 for $z_0=0$. Note $s_1(0)=s_2(0)=i\infty$. Apply Theorem 4.1 (iv). Then we have the composite map:

$$\{C(g)\in Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})\,|\,C(s_1)=C(s_2)\}\subset Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})\to\frac{Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})}{B^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})}$$

$=H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})\cong S_{w+2}(\Gamma)=S_{w+2}(\Gamma',\chi)$ where $\to$ denotes the canonical reduction map and ($\chi=\chi_0$ if $2\,|\,w$, and $\chi=\chi_1$ if $w$ is odd). From Theorem 4.1 and Lemma 2 we have

**Corollary of Theorem 4.1 and Lemma 2.** *The above composite map is surjective.*

We have

**Lemma 6.** *Let* $R$ *be either* $\mathbb{R}$ *or* $\mathbb{Z}$. *The* $R$*-linear map*

$\{C(g)\in Z_P^1(SL(2,\mathbb{Z}),\eta_w,R)\,|\,C(s_1)=C(s_2)\}\to$

$\{\mathbf{x} \in R^{(w+1)m} \mid (\mathbf{1}+\eta_w(s_1))\mathbf{x} = (\mathbf{1}+\eta_w(s_2)+\eta_w(s_2)^2)\mathbf{x} = \mathbf{0}\}$, $C(s_1) \mapsto C(s_1)$ *is bijective.*

Proof of Lemma 6. Injectivity of this map is trivial. We shall show surjectivity. Assume that $\mathbf{x} \in \mathbb{R}^{(w+1)m}$ satisfies $(\mathbf{1}+\eta_w(s_1))\mathbf{x} = (\mathbf{1}+\eta_w(s_2)+\eta_w(s_2)^2)\mathbf{x} = \mathbf{0}$. If $C$ is a crossed homomorphism with $\mathbf{x} = C(s_1) = C(s_2)$, then the value of $C(g)$ at each $g \in SL(2,\mathbb{Z})$ is determined by $SL(2,\mathbb{Z}) = < s_1, s_2 >$ and the map $g \mapsto C(g)$ becomes an element of $Z^1(SL(2,\mathbb{Z}),\eta_w,R)$. Further we shall show $C \in Z_P^1(SL(2,\mathbb{Z}),\eta_w,R)$.

Put $k = s_1^{-1}s_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in P$. Note $C(\mathbf{1}_2) = C(-\mathbf{1}_2) = 0$ and $\eta_w(-\mathbf{1}_2) = 1$. We have

$C(k) = C(s_1 s_2) = C(s_1) + \eta_w(s_1)C(s_2) = (\mathbf{1}+\eta_w(s_1))\mathbf{x} = \mathbf{0}$. From $k^{-1}k = \mathbf{1}_2$, we have also $C(k^{-1}) = 0$. Hence we have $C(k^n) = 0$ for any $n \in \mathbb{Z}$. Let $\gamma \in P$. Then we have $\gamma = \pm g k^n g^{-1}$ with some $g \in SL(2,\mathbb{Z})$ and some $n \in \mathbb{Z}$. We compute as follows.

$$C(gk^n g^{-1}) = C(g) + \eta_w(g)C(k^n g^{-1}) = C(g) + \eta_w(g)(C(k^n) + \eta_w(k^n)C(g^{-1}))$$
$$= C(g) + \eta_w(g)(C(k^n) + \eta_w(k^n)(-\eta_w(g)^{-1}C(g))) = C(g) - \eta_w(gk^n g^{-1})C(g)$$
$$= (1 - \eta_w(gk^n g^{-1}))C(g) \in (1 - \eta_w(gk^n g^{-1}))R^{(w+1)m}.$$

$$C(-gk^n g^{-1}) = C(gk^n g^{-1}) = (1 - \eta_w(gk^n g^{-1}))C(g)$$
$$= (1 - \eta_w(-gk^n g^{-1}))C(g) \in (1 - \eta_w(-gk^n g^{-1}))R^{(w+1)m}.$$

Lemma 6 is proven.

We have $ts_2t = s_1 s_2^2 s_1$ and $ts_1t = -s_1 = s_1^{-1}$. We have also $ts_2^2 t = s_1 s_2 s_1$.

**Lemma 7.** *Let* $R$ *be either* $\mathbb{R}$ *or* $\mathbb{Z}$. *We put*

$\Omega(R) = \{\mathbf{x} \in R^{(w+1)m} \mid (\mathbf{1}+\eta_w(s_1))\mathbf{x} = (\mathbf{1}+\eta_w(s_2)+\eta_w(s_2)^2)\mathbf{x} = \mathbf{0}\}$ *and*

$\Omega_t(R) = \{\mathbf{x} \in R^{(w+1)m} \mid (\mathbf{1}+\eta_w(ts_1t))\mathbf{x} = (\mathbf{1}+\eta_w(ts_2t)+\eta_w(ts_2t)^2)\mathbf{x} = \mathbf{0}\}$ *for short.*

*Then we have* $\Omega(R) = \Omega_t(R)$.

Proof of Lemma 7. We have $\mathbf{1} + \eta_w(ts_2t) + \eta_w(ts_2t)^2 = \eta_w(s_1)(\mathbf{1} + \eta_w(s_2) + \eta_w(s_2)^2)\eta_w(s_1)$

and $\eta_w(ts_1t)(\mathbf{1} + \eta_w(ts_2t) + \eta_w(ts_2t)^2)\eta_w(ts_1t) = \mathbf{1} + \eta_w(s_2) + \eta_w(s_2)^2$.

Assume first $\mathbf{x} \in \Omega(R)$. Then we have

$(\mathbf{1} + \eta_w(ts_1t))\mathbf{x} = (\mathbf{1} + \eta_w(s_1^{-1}))\mathbf{x} = \eta_w(s_1^{-1})(\mathbf{1} + \eta_w(s_1))\mathbf{x} = \mathbf{0}$ and

$$(\mathbf{1} + \eta_w(ts_2t) + \eta_w(ts_2t)^2)\mathbf{x} = \eta_w(s_1)(\mathbf{1} + \eta_w(s_2) + \eta_w(s_2)^2)(\eta_w(s_1))(\mathbf{x})$$
$$= \eta_w(s_1)(\mathbf{1} + \eta_w(s_2) + \eta_w(s_2)^2)(-\mathbf{x}) = \mathbf{0}.$$

Hence $\mathbf{x} \in \Omega_t(R)$.

Now assume $\mathbf{x} \in \Omega_t(R)$. Then we have

$\mathbf{0} = \eta_w(s_1)(\mathbf{1} + \eta_w(ts_1t))\mathbf{x} = \eta_w(s_1)(\mathbf{1} + \eta_w(s_1^{-1}))\mathbf{x} = (\mathbf{1} + \eta_w(s_1))\mathbf{x}$ and

$$\mathbf{0} = (\mathbf{1} + \eta_w(ts_2t) + \eta_w(ts_2t)^2)\mathbf{x} = \eta_w(s_1)(\mathbf{1} + \eta_w(s_2) + \eta_w(s_2)^2)(\eta_w(s_1))(\mathbf{x})$$
$$= \eta_w(s_1)(\mathbf{1} + \eta_w(s_2) + \eta_w(s_2)^2)(-\mathbf{x}).$$

Hence $\mathbf{x} \in \Omega(R)$.
Lemma 7 is proven.

**Lemma 8.** *Let* $R$ *be either* $\mathbb{R}$ *or* $\mathbb{Z}$. *We put*

$B(R) = \{\mathbf{z} \in R^{(w+1)m} \mid \exists \mathbf{y}(\mathbf{y} \in R^{(w+1)m}$ and $\mathbf{z} = (\mathbf{1} - \eta_w(s_1))\mathbf{y} = (\mathbf{1} - \eta_w(s_2))\mathbf{y})\}$ *and*

$B_t(R) = \{\mathbf{z} \in R^{(w+1)m} \mid \exists \mathbf{y}(\mathbf{y} \in R^{(w+1)m}$ and $\mathbf{z} = (\mathbf{1} - \eta_w(\mathbf{t}s_1\mathbf{t}))\mathbf{y} = (\mathbf{1} - \eta_w(\mathbf{t}s_2\mathbf{t}))\mathbf{y})\}$ *for short. Then we have* $B(R) = B_t(R)$.

Proof of Lemma 8. We have

$$(\mathbf{1}-\eta_w(s_1))\mathbf{y}=(\mathbf{1}-\eta_w(s_2))\mathbf{y}\Leftrightarrow\eta_w(s_1^{-1}s_2)\mathbf{y}=\mathbf{y}\Leftrightarrow\eta_w\left(\begin{pmatrix}1&0\\-1&1\end{pmatrix}\right)\mathbf{y}=\mathbf{y}\quad\text{and}$$

$$(\mathbf{1}-\eta_w(ts_1t))\mathbf{y}=(\mathbf{1}-\eta_w(ts_2t))\mathbf{y}\Leftrightarrow\eta_w(ts_1^{-1}s_2t)\mathbf{y}=\mathbf{y}\Leftrightarrow\eta_w(ts_2^{-1}s_1t)\mathbf{y}=\mathbf{y}\Leftrightarrow\eta_w\left(\begin{pmatrix}1&0\\-1&1\end{pmatrix}\right)\mathbf{y}=\mathbf{y}.$$

Lemma 8 is proven.

We write $s_1^{-1}s_2=\begin{pmatrix}1&0\\-1&1\end{pmatrix}=k$ for short. Then $k\in P$.

**Theorem 9.** *The image of the composite map given by*

$$\{C(g)\in Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})\mid C(s_1)=C(s_2)\}\subset Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})\subset Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$$
$$\xrightarrow{\text{can.reduction}}\frac{Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})}{B^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})}=H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$$

*is equal to* $j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z}))$.

Proof of Theorem 9. That the image $\subset j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z}))$ follows directly from the definitions of $H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})$ and $H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$.

We shall show the converse of this, namely, (the image $\supset H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})$).

Let $u\in H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})$. By Theorem 4.1 (iv) there exists a cusp form $f$ in $S_{w+2}(\Gamma)$ such that the cohomology class in $H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ of the cocycle $C^*$ :

$$C^*(g)=\operatorname{Re}\int_0^{g(0)}D(f)\qquad(g\in SL(2,\mathbb{Z}))$$

is equal to $j_1(u)$. Since $j_1(u)\in j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z}))$, there exist vectors $\mathbf{X}\in\mathbb{R}^{(w+1)m}$ and $\mathbf{T}\in\mathbb{Z}^{(w+1)m}$ such that

(9.1) $\qquad C^*(g)+(\mathbf{1}-\eta_w(g))\mathbf{X}\in\mathbb{Z}^{(w+1)m}\quad$ for all $g\in SL(2,\mathbb{Z})$

and such that

(9.2) $$C^*(k)+(\mathbf{1}-\eta_w(k))\mathbf{X}=(1-\eta_w(k))\mathbf{T}$$

since $k\in P$. Note $C^*(k)=0$ since $k(0)=0$. Hence $(\mathbf{1}-\eta_w(k))(\mathbf{X}-\mathbf{T})=\mathbf{0}$. Put

$C(g)=C^*(g)+(\mathbf{1}-\eta_w(g))(\mathbf{X}-\mathbf{T})$ for all $g\in SL(2,\mathbb{Z})$. Since $\mathbf{T}\in\mathbb{Z}^{(w+1)m}$, from (9.1) and (9.2), we have

$$C(g)\in\mathbb{Z}^{(w+1)m} \quad \text{for all } g\in SL(2,\mathbb{Z}),$$

and

$$C(k)=C^*(k)+(\mathbf{1}-\eta_w(k))(\mathbf{X}-\mathbf{T})=\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Hence the $C$ is a cocycle in $Z_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})$ since any parabolic element of $SL(2,\mathbb{Z})$ is conjugate to $\pm k^m$ in $SL(2,\mathbb{Z})$ for some integer $m\in\mathbb{Z}$. Note $C^*(s_1)=C^*(s_2)$. We compute as follows.

$$C(s_1)-C(s_2)=C^*(s_1)-C^*(s_2)+(\eta_w(s_2)-\eta_w(s_1))(\mathbf{X}-\mathbf{T})$$
$$=-\eta_w(s_1)(\mathbf{1}-\eta_w(k))(\mathbf{X}-\mathbf{T})=\mathbf{0}.$$

It is easy to see that the cohomology class $\in H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ of the cocycle $C$ is equal to $j_1(u)$. Theorem 9 is proven.

**Theorem 10.** *We use the notations given in Lemmas 6, 7 and 8 above. Let* $R=\mathbb{R}$. *The linear map* $\psi$ *from* $\Omega(\mathbb{R})$ *to* $H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$ *given by* $\mathbf{x}\mapsto$ *(cohomology class of cocycle determined by* $s_1\mapsto\mathbf{x}$ *and* $s_2\mapsto\mathbf{x}$*) is surjective. Hence it induces the linear isomorphism* $\Psi:\dfrac{\Omega(\mathbb{R})}{B(\mathbb{R})}\to H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R})$.

Proof of Theorem 10. This theorem follows from Lemma 4.1 and Lemmas 2, 6, 7 and 8.

**Theorem 11.** *We use the notations in Lemmas 6, 7 and 8.*

(i) *If* $\mathbf{x} \in \Omega(R)$, *then* $\eta_w(\mathbf{t})\mathbf{x} \in \Omega(R)$. *The linear map :* $\Omega(R) \to \Omega(R)$, $\mathbf{x} \mapsto \eta_w(\mathbf{t})\mathbf{x}$ *is an isomorphism.*

(ii) *If* $\mathbf{x} \in B(R)$, *then* $\eta_w(\mathbf{t})\mathbf{x} \in B(R)$. *The linear map :* $B(R) \to B(R)$, $\mathbf{x} \mapsto \eta_w(\mathbf{t})\mathbf{x}$ *is an isomorphism.*

(iii) *Let* $R = \mathbb{R}$. *The following diagram is commutative.*

$$\begin{array}{ccccc} \Omega(\mathbb{R}) & \xrightarrow{\psi} & H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R}) & \xrightarrow{\varphi^{-1}\circ \mathbf{sh}_P\mathbb{R}} & S_{w+2}(\Gamma) \\ \downarrow L & & & & \downarrow \varepsilon \\ \Omega(\mathbb{R}) & \xrightarrow{\psi} & H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R}) & \xrightarrow{\varphi^{-1}\circ \mathbf{sh}_P\mathbb{R}} & S_{w+2}(\Gamma) \end{array}$$

*where* $L(\mathbf{x}) = \eta_w(\mathbf{t})\mathbf{x}$ *for any* $\mathbf{x} \in \Omega(\mathbb{R})$, *and* $(\varepsilon(f))(z) = \overline{f(-\bar{z})}$ *for any* $f \in S_{w+2}(\Gamma)$.

Proof of Theorem 11. Note $\eta_w(\mathbf{t})$ is invertible in $GL(2,\mathbb{Z})$.

(i) We have $\mathbf{x} \in \Omega(\mathbf{R}) \Leftrightarrow (\mathbf{1}+\eta_w(s_1))\mathbf{x} = (\mathbf{1}+\eta_w(s_2)+\eta_w(s_2)^2)\mathbf{x} = \mathbf{0} \Leftrightarrow$

$$\eta_w(\mathbf{t})(\mathbf{1}+\eta_w(s_1))\eta_w(\mathbf{t})^{-1}\eta_w(\mathbf{t})\mathbf{x} = \eta_w(\mathbf{t})(\mathbf{1}+\eta_w(s_2)+\eta_w(s_2)^2)\eta_w(\mathbf{t})^{-1}\eta_w(\mathbf{t})\mathbf{x} = \mathbf{0}$$

$\Leftrightarrow \eta_w(\mathbf{t})\mathbf{x} = \mathbf{0}$ by Lemma 7.

(ii) We have $\mathbf{z} \in B(\mathbf{R}) \Leftrightarrow \mathbf{z} = (\mathbf{1}-\eta_w(s_1))\mathbf{x} = (\mathbf{1}-\eta_w(s_2))\mathbf{x} \Leftrightarrow$

$$\eta_w(\mathbf{t})\mathbf{z} = \eta_w(\mathbf{t})(\mathbf{1}-\eta_w(s_1))\eta_w(\mathbf{t})^{-1}\eta_w(\mathbf{t})\mathbf{x} = \eta_w(\mathbf{t})(\mathbf{1}-\eta_w(s_2))\eta_w(\mathbf{t})^{-1}\eta_w(\mathbf{t})\mathbf{x}$$

$\Leftrightarrow \eta_w(\mathbf{t})\mathbf{z} \in B(R)$ by Lemma 8.

(iii) Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$ and let $f \in S_{w+2}(\Gamma)$. By direct computation we have

$$\overline{(f(z)z^k dz)\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}} = (-1)^{k+1}((\varepsilon(f))(u)u^k du)\circ\begin{pmatrix}\alpha & -\beta\\ -\gamma & \delta\end{pmatrix}$$ . Hence we have

$$\mathrm{Re}\int_0^{i\infty}(f(z)z^k dz)\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \mathrm{Re}\int_0^{i\infty}\overline{(f(z)z^k dz)\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}}$$

$$= (-1)^{k+1}\,\mathrm{Re}\int_0^{i\infty}((\varepsilon(f))(u)u^k du)\circ\begin{pmatrix}\alpha & -\beta\\ -\gamma & \delta\end{pmatrix}$$

and $\mathrm{Re}\int_0^{i\infty}((\varepsilon(f))(z)z^k dz)\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = (-1)^{k+1}\,\mathrm{Re}\int_0^{i\infty}(f(u)u^k du)\circ\begin{pmatrix}\alpha & -\beta\\ -\gamma & \delta\end{pmatrix}$.

Namely we have $\mathrm{Re}\int_0^{i\infty} D(\varepsilon(f)) = \eta_w(\mathbf{t})\,\mathrm{Re}\int_0^{i\infty} D(f)$.

Note that $\sigma\in SL(2,\mathbb{Z}) \mapsto \mathrm{Re}\int_0^{\sigma(0)} D(F)$ is a parabolic cocycle with

$$\mathrm{Re}\int_0^{i\infty} D(F) = \mathrm{Re}\int_0^{s_1(0)} D(F) = \mathrm{Re}\int_0^{s_2(0)} D(F).$$

Theorem 11 is proven.

**Definitions.**

$\Omega(\mathbb{R})^+ = \{\mathbf{x}\in\Omega(\mathbb{R}) \mid \eta_w(\mathbf{t})\mathbf{x} = \mathbf{x}\}$ and $B(\mathbb{R})^+ = B(\mathbb{R})\cap\Omega(\mathbb{R})^+$;

$\Omega(\mathbb{R})^- = \{\mathbf{x}\in\Omega(\mathbb{R}) \mid \eta_w(\mathbf{t})\mathbf{x} = -\mathbf{x}\}$ and $B(\mathbb{R})^- = B(\mathbb{R})\cap\Omega(\mathbb{R})^-$;

$\Omega(\mathbb{Z}) = \mathbb{Z}^{(w+1)m}\cap\Omega(\mathbb{R})$ and $B(\mathbb{Z}) = B(\mathbb{R})\cap\Omega(\mathbb{Z})$.

We have already

$\mathrm{Re}\int_0^{i\infty} D(F)\in\Omega(\mathbb{R})^+$ for any $F\in S_{w+2}^{\mathbb{R}}(\Gamma)$ , and

$\mathrm{Re}\int_0^{i\infty} D(F)\in\Omega(\mathbb{R})^-$ for any $F\in\sqrt{-1}S_{w+2}^{\mathbb{R}}(\Gamma)$.

From Theorems 10 and 11 we have the isomorphism :

$$\frac{\Omega(\mathbb{R})}{B(\mathbb{R})} \xrightarrow{\Psi} H^1_P(SL(2,\mathbb{Z}),\eta_w,\mathbb{R}) \xrightarrow{\varphi^{-1}\circ \mathrm{sh}_P\mathbb{R}} S_{w+2}(\Gamma). \qquad (11.1)$$

**Corollary of Theorem 11.**

(i) *We have not only* $\mathrm{Image}(\varphi^{-1}\circ \mathrm{sh}_P\mathbb{R}\circ\psi|_{\Omega(\mathbb{R})^+})\subset S^{\mathbb{R}}_{w+2}(\Gamma)$ *but also*

$$\mathrm{Image}(\varphi^{-1}\circ sh_P\mathbb{R}\circ\psi|_{\Omega(\mathbb{R})^+}) = S^{\mathbb{R}}_{w+2}(\Gamma).$$

(ii) *We have not only* $\mathrm{Image}(\varphi^{-1}\circ \mathrm{sh}_P\mathbb{R}\circ\psi|_{\Omega(\mathbb{R})^-})\subset \sqrt{-1}S^{\mathbb{R}}_{w+2}(\Gamma)$ *but also*

$$\mathrm{Image}(\varphi^{-1}\circ sh_P\mathbb{R}\circ\psi|_{\Omega(\mathbb{R})^-}) = \sqrt{-1}S^{\mathbb{R}}_{w+2}(\Gamma).$$

(iii) *Restricting the domain* $\frac{\Omega(\mathbb{R})}{B(\mathbb{R})}$ *of* $\varphi^{-1}\circ \mathrm{sh}_P\mathbb{R}\circ\Psi$ *to* $\frac{\Omega(\mathbb{R})^+}{B(\mathbb{R})^+}$ *we have the isomorphism from* $\frac{\Omega(\mathbb{R})^+}{B(\mathbb{R})^+}$ *to* $S^{\mathbb{R}}_{w+2}(\Gamma)$.

(iv) *Restricting the domain* $\frac{\Omega(\mathbb{R})}{B(\mathbb{R})}$ *of* $\varphi^{-1}\circ \mathrm{sh}_P\mathbb{R}\circ\Psi$ *to* $\frac{\Omega(\mathbb{R})^-}{B(\mathbb{R})^-}$ *we have the isomorphism from* $\frac{\Omega(\mathbb{R})^-}{B(\mathbb{R})^-}$ *to* $\sqrt{-1}S^{\mathbb{R}}_{w+2}(\Gamma)$.

**Remark.**

In [9], the only case of $\Gamma = SL(2,\mathbb{Z})$ of the above Corollary (iii) and (iv) of Theorem 11 is announced without any proof. In [8, pp.89-96 ] Lang gives his proof of only $\Gamma = SL(2,\mathbb{Z})$ case of (11.1). The proof of Lang's is completely different from ours given in [5], [6], [7] and our present paper. Lang's proof in [8, pp. 89-96] is complicated and does not seem to be extended to cusp forms on arbitrary congruence subgroups of $SL(2,\mathbb{Z})$. Lang announces [8, p.72, Theorem 1.1] which is equivalent to $\Gamma = SL(2,\mathbb{Z})$ case of (iii) and (iv) of the above Corollary of our Theorem 11. Lang gives no proof of [8, p.72, Theorem 1.1] even for $\Gamma = SL(2,\mathbb{Z})$ although he writes ``This theorem will be proved in the next chapter" in [8, p. 72, Line 7 from the bottom of the page].
In [10] the only case of $\Gamma = SL(2,\mathbb{Z})$ and $w+2=12$ of Corollary (iii) and (iv) of Theorem 11 is computed directly by hand.

Through $\Omega(\mathbb{Z}) \subset \Omega(\mathbb{R})$, we have the injection $\iota : \dfrac{\Omega(\mathbb{Z})}{B(\mathbb{Z})} \to \dfrac{\Omega(\mathbb{R})}{B(\mathbb{R})}$. We consider the following composite map, which is injective.

$$\frac{\Omega(\mathbb{Z})}{B(\mathbb{Z})} \xrightarrow{\iota} \frac{\Omega(\mathbb{R})}{B(\mathbb{R})} \xrightarrow{\Psi} H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{R}) \xrightarrow{\varphi^{-1}\circ \mathrm{sh}_P\mathbb{R}} S_{w+2}(\Gamma). \qquad (11.2)$$

By Theorem 9, the image of $\Psi \circ \iota$ is $j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z}))$. Hence the image of $\varphi^{-1} \circ \mathrm{sh}_P\mathbb{R} \circ \Psi \circ \iota$ is equal to $j_2(H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{Z}))$ where $\chi = \chi_0$ if $2 \mid w$, and $\chi = \chi_1$ if $w$ is odd. Recall Proposition 1. Namely we have that

$\varphi^{-1} \circ \mathrm{sh}_P\mathbb{R}(j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})))$ is stable by $\varepsilon$.

Summing up the above results we give

**Proposition 12.** *We have the isomorphism*

$$\frac{\Omega(\mathbb{Z})}{B(\mathbb{Z})} \cong j_1(H_P^1(SL(2,\mathbb{Z}),\eta_w,\mathbb{Z})) \cong j_2(H^1_{P\cap\Gamma'}(\Gamma',\chi\otimes\rho_w|_{\Gamma'},\mathbb{Z}))$$

*where* $\chi = \chi_0$ *if* $2 \mid w$, *and* $\chi = \chi_1$ *if* $w$ *is odd.*

If $B(\mathbb{R})^+ = \{\mathbf{0}\}$, it is possible to see the action of Hecke operators $T_{w+2}(n)$ on $\Omega(\mathbb{Z})^+$ or $\Omega(\mathbb{R})^+$ by using continued fractions of rational numbers, cf. [7], [4], [3], [2], [1].

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# Appendix

## Corrections of some misprints in [6]

p.14, L.1 from the bottom of the page: replace `$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$' by `$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$'.

p.15, L.1 from the top of the page: replace `$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$' by `$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$'.

p.16, L.10 from the bottom of the page: insert `Re' in front of the integral.
p.9, L.9-10 from the top of the page: We had better replace

` $\rho_w\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)dz_w$

$={}^t((cz+d)^w,(cz+d)^w(az+b),(cz+d)^{w-2}(az+b)^2,\dots,(az+b)^w)dz$ '

by

` $\rho_w\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)dz_w$

$=(\alpha\delta-\beta\gamma)\ {}^t((\gamma z+\delta)^w,(\gamma z+\delta)^w(\alpha z+\beta),(\gamma z+\delta)^{w-2}(\alpha z+\beta)^2,\dots,(\alpha z+\beta)^w)dz$ '

in order to distinguish the letter $d$ in the matrix $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ from the letter $d$ in the differential forms $dz_w$ and $dz$ on the complex upper half pane.