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CIRCULAR EMBEDDINGS OF 2-COMPONENT LINKS AND SPATIAL GRAPHS

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Abstract. A circular spatial embedding of a graph into the 3-dimensional Euclidean space $\mathbb{R}^3$ is an embedding which consists of round arcs. We define the circular number of a spatial graph as the minimal number of round arcs in $\mathbb{R}^3$ among such embeddings of the spatial graph. In this paper, we investigate the circular number for 2-component links, theta-curves and handcuff graphs.

Key words: Knots, Links, Spatial graphs, Circular number.

1. Introduction

A *graph* G is a 1-dimensional simplicial complex. We call an embedding $f : G \longrightarrow \mathbb{R}^3$ of a graph G in $\mathbb{R}^3(= \{(x, y, z) \mid x, y, z \in \mathbb{R}\})$ a *spatial embedding* of G and a graph $f(G)$ embedded in $\mathbb{R}^3$ a *spatial graph*. A *cycle* C in a graph G is a cyclic sequence $\{v_0, \cdots, v_{n-1}\}$ of distinct vertices of G such that any two consecutive vertices v_i and v_{i+1}, i taken modulo n, are joined by an edge v_iv_{i+1}. If one embeds G in $\mathbb{R}^3$ by a spatial embedding $f : G \longrightarrow \mathbb{R}^3$, then the image of any cycle in G is a *knot*, and any disjoint union of cycles in G forms a *link* in $\mathbb{R}^3$. A spatial embedding of a graph which maps each edge to a round arc in $\mathbb{R}^3$ is called a *circular spatial embedding*. For a spatial graph $\hat{G}$, we define the *circular number* $Circ(\hat{G})$ as the minimal number n such that $\hat{G}$ has a circular spatial embedding with n round arcs. Clearly, the circular number is an invariant of a spatial graph. It is known that every graph has a circular spatial embedding and a knot has circular number three if and only if the knot is a *trefoil knot*, and the *figure-eight knot* has circular number four [1]. A spatial graph is *trivial* if it has an embedding in the plane.

Theorem 1.1.
(1) *A 2-component link L is either the trivial link or the Hopf link if and only if $Circ(L) = 2$.*
(2) *There is no 2-component link L with $Circ(L) = 3$.*
(3) *If a 2-component link L has circular number four, then each component of L is either a trefoil knot or the trivial knot.*

A *theta-graph* is a (plane) graph with two vertices and three edges where each edge connects the two vertices. A *theta-curve* is a spatial embedding of a theta-graph.

Proposition 1.2.
(1) *A theta-curve T is trivial if and only if $Circ(T) = 2$.*
(2) *There exists a (non-trivial) theta-curve T with $Circ(T) = 3$.*

A *handcuff graph* is a spatial embedding of a (plane) graph which is formed by two disjoint loops connected by an edge.

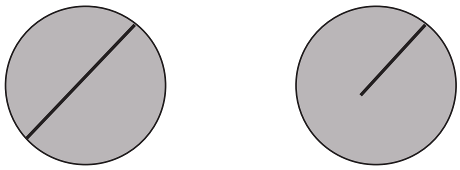
Proposition 1.3.
(1) *If H is a handcuff graph with $Circ(H) = 3$, then H is the trivial link or the Hopf link with an unknotted edge connecting the two components.*
(2) *There exists a non-trivial Handcuff graph H with $Circ(H) = 3$.*

This paper is organized as follows. In Section 2, we shall prove Theorems 1.1 and Propositions 1.2 and 1.3. In Section 3, we shall give some examples.

2. Proofs

Definition 2.1. (1) Let G be a graph. A spatial embedding $f : G \longrightarrow \mathbb{R}^3$ is called *a circular spatial embedding* if there exists a finite number of round circles $\ell_1, \cdots, \ell_n$ in $\mathbb{R}^3$ such that $f(G) \subset \cup_{i=1}^{n} \ell_i$.
(2) For a spatial graph $\hat{G}$, let $Circ(\hat{G}) = \min\{n \in \mathbb{N} \mid \hat{G}$ consists of n round arcs$\}$. We call $Circ(\hat{G})$ the *circular number* of $\hat{G}$. (We assume that a round circle is also a round arc.)

Proof of Theorem 1.1. First, we show the statement (1). It follows from the definition of the circular number that if L is the trivial link or the Hopf link, then $Circ(L) = 2$. Suppose $Circ(L) = 2$. Then by the assumption, L can be composed with two round circles in $\mathbb{R}^3$. We denote the two disks which is bounded by the two circles by D_1 and D_2. Then we may assume that D_1 and D_2 intersect each other as in Figure 1. (The lines in the disks describe the intersection.)



(a) (b)

Figure 1

If one of the disks has intersection as in Figure 1(a), then the intersection is canceled by an isotopy of $\mathbb{R}^3$ and we have the trivial link. If both disks have intersection as in Figure 1(b), then the link is the Hopf link. Next we show the statement (2). Suppose that L is a 2-component link with $Circ(L) = 3$. Then one of the components of L consists of a round circle and the other one is composed with two round arcs. We denote them by C and C' respectively. Note that C' bounds a disk D as in Figure 2.

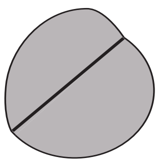


Figure 2

We may assume that D meets C in a finite number of points. In fact, we know that $D \cap C$ is either empty or consists of n points where $1 \leq n \leq 4$. Then it is easily seen that we have the trivial link or the Hopf link. However their circular numbers are less than three. This is contradiction. Finally, we show the statement (3). If $Circ(L) = 4$ then both components of L consists of two round arcs or one of the components consists of a round circle and the other consists of three round arcs. In the former case, all components are trivial. In the latter case, we have the trivial knot and either a trefoil knot or the trivial knot as the components by a result in [1].

Proof of Proposition 1.2. If a theta curve T is trivial, then it is easily seen that $Circ(T) = 2$ since we need at least two round arcs to yield a theta curve. Conversely, if $Circ(T) = 2$, then T consists of two round arcs and one of them must be a round circle. Then we know that the circle with a round arc can be embedded in the plane generically. Thus the statement (1) holds. To show the statement (2), we consider a theta-curve T as in Figure 3(a). It has a circular embedding as shown in Figure 3 (b). Then we know that $Circ(T) \leq 3$. By the statement (1), we also know that $Circ(T) \geq 3$.

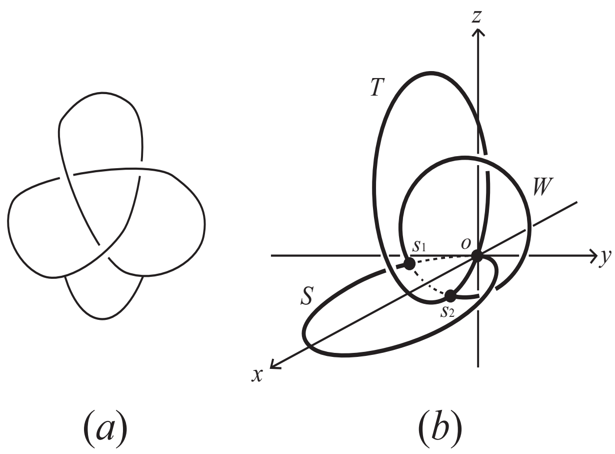


Figure 3

Proof of Proposition 1.3. First we show the statement (1), if $Circ(H) = 3$, then H has two round circles and a round arc. The two circles realize either the trivial link or the Hopf link where the two circles are connected by a round arc. To show the statement (2), we consider the Handcuff graph $\hat{H}$ as shown in Figure 4.

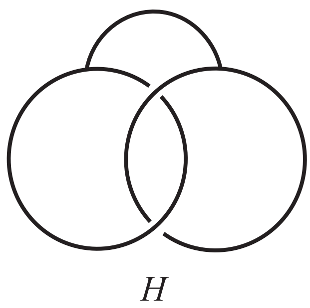


Figure 4

By the definition of the circular number, we know that $Circ(\hat{H}) \leq 3$. We need at least three round arcs to compose a Handcuff graph since it has trivalent vertices. So we know that $Circ(\hat{H}) \geq 3$. Since it has the Hopf link as its cycles, we know that $\hat{H}$ is non-trivial. This completes the proof.

3. Examples

Example 1. Since the figure eight knot have circular numbers four [1], the theta curves as in Figure 5(a) are four. In fact, it has a circular embedding with four round arcs as shown in Figure 5(b).

Example 2. The circular numbers of the spacial handcuff graph as shown in Figure 6(a) and (b) are less than or equal to four. However we do not know if the circular numbers are equal to four or not.

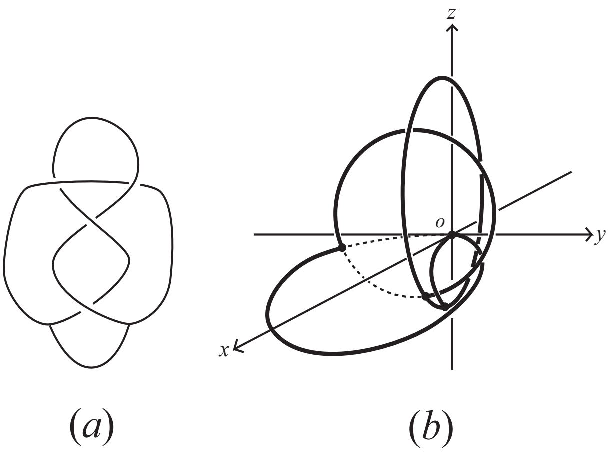


Figure 5

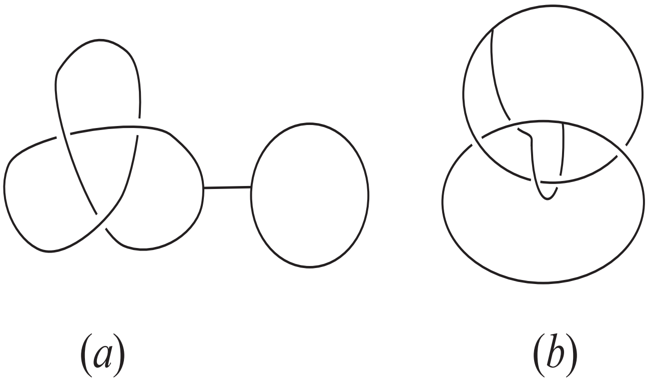


Figure 6

References

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