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A KNOT INVARIANT DERIVED FROM GAMMA MOVES

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Abstract. Any knot can be changed into either the unknot or the trefoil knot by a finite sequence of gamma moves. We define the gamma index of a knot as the minimum number of moves among all such sequences. We show that the gamma indices of the knots $4_1, 5_2, 6_1$ and 6_3 are equal to one and the gamma index of the connected sum of the trefoil knot and its mirror image is equal to two.

Key words: Knot; Gamma move.

1. Introduction

A *knot* is an embedding of a circle in the oriented 3-dimensional sphere S^3. Two knots are said to be *equivalent* if they represent ambient isotopic embeddings.

We define *gamma moves* to be changes in a diagram of a knot as in Figure 1. We call two knots *gamma equivalent* if there exists a sequence of gamma moves that take us from the one knot to the other, where we can arrange the diagram of the knot any way that we want after each gamma move. Kauffman showed that every knot is gamma equivalent to either the unknot or the trefoil knot [2].

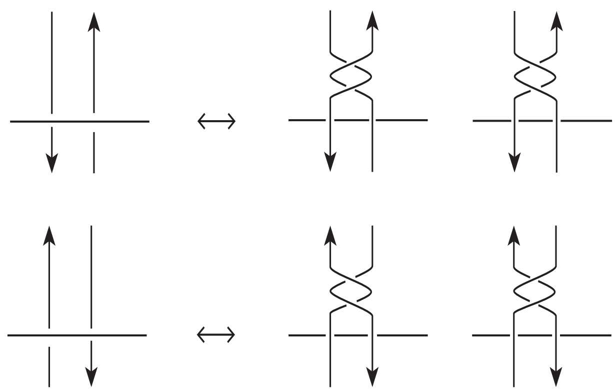

Figure 1

We define the *gamma index* of a knot K as the minimal number of gamma moves needed to convert the knot into either the unknot or the trefoil knot where the minimum is taken over all possible diagrams of the knot. We denote it by $\Gamma(K)$.

Theorem 1.1. *For any knot K in $\{4_1, 5_2, 6_1, 6_3\}$ as in Figure 2, we have $\Gamma(K) = 1$.*

Let $\hat{K}$ be the connected sum of the trefoil knot and its mirror image as in Figure 3. Then let K_n be the connected sum of n $\hat{K}$'s.

Theorem 1.2. *We have $\frac{2}{3}n \leq \Gamma(K_n) \leq n$ for any positive integer n. In particular, $\Gamma(K_2) = 2$.*

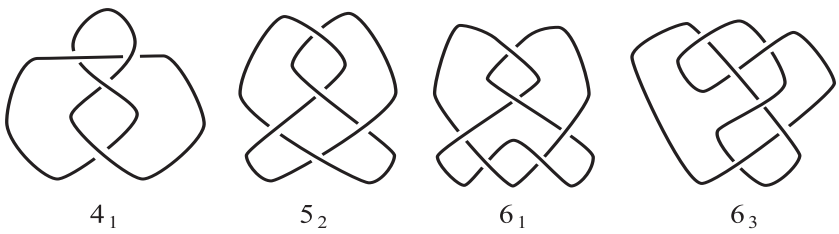

Figure 2

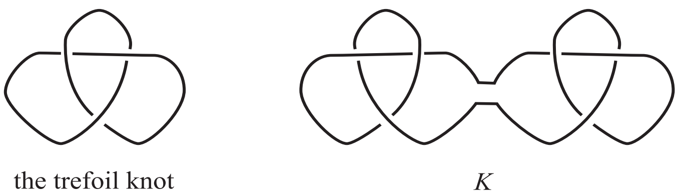

Figure 3

2. Proof of Theorem

Lemma 2.1. *Each of the following moves is obtained from a single gamma move.*

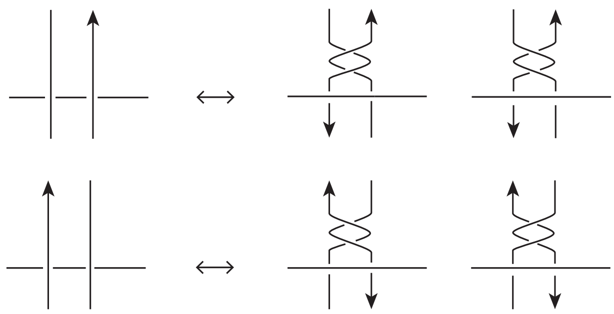

Figure 4

Proof. The result is obtained as shown in Figure 5.

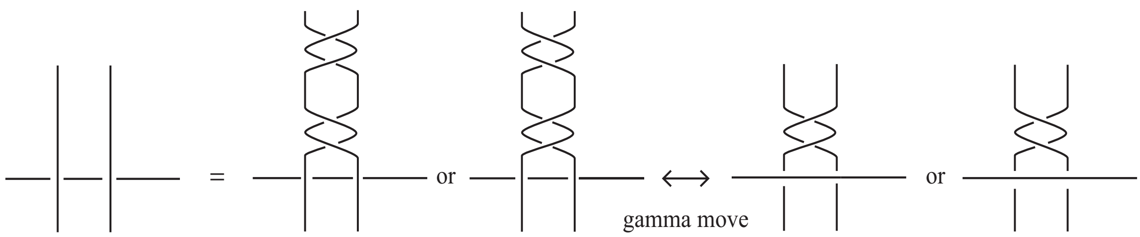

Figure 5

□

For a knot K, we define the Arf invariant $Arf(K)$ [1] [5] of K by $Arf(K) = a_2(K)$ (mod 2), where a_2 is the second coefficient of the *Conway polynomial* of K. We need the following result.

Proposition 2.2. (Kauffman)[2] *Two knots are gamma-equivalent if and only if the Arf invariants are equal.*

By the following proposition, we know that there is a knot of arbitrarily large gamma index.

Proposition 2.3. *For a knot K, $u(K) \leq 3\Gamma(K)+1$ if $Arf(K) = 1$ and $u(K) \leq 3\Gamma(K)$ if $Arf(K) = 0$, where $u(K)$ is the unknotting number of K.*

Proof. A gamma move is realized by applying three crossing changes and the unknotting number of the trefoil knot is equal to one. Thus the inequalities are obtained from Proposition 2.2.

□

Proof of Theorem 1.1. The theorem is shown by using the definition of gamma moves and lemma 2.1 as follows. We can show $\Gamma(4_1) = 1$ as shown in Figure 6.

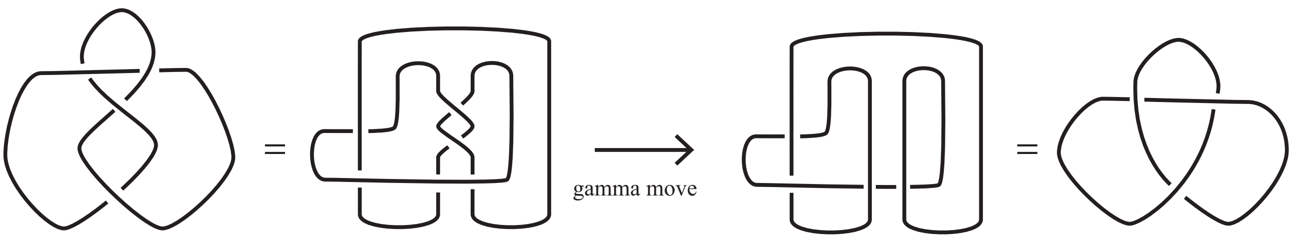

FIGURE 6

We have $\Gamma(5_2) \leq 1$ as shown in Figure 7.

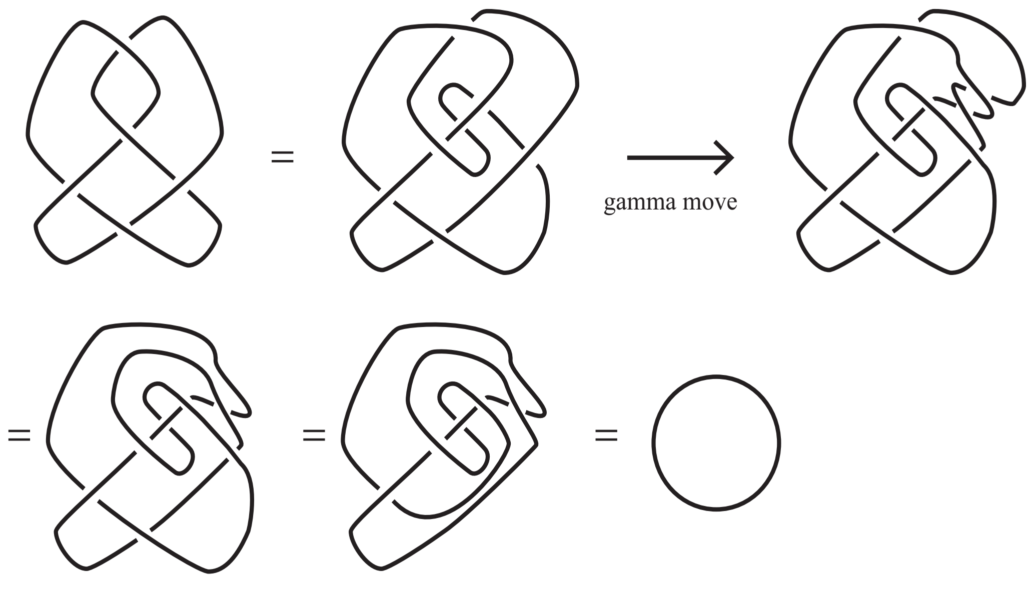

FIGURE 7

We have $\Gamma(6_1) \leq 1$ as shown in Figure 8.

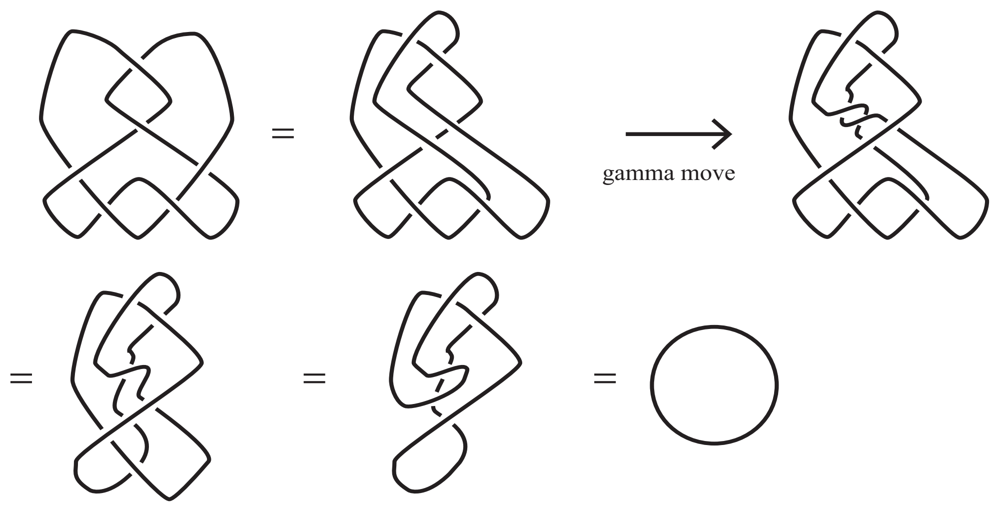

Figure 8

We have $\Gamma(6_3) \leq 1$ as shown in Figure 9.

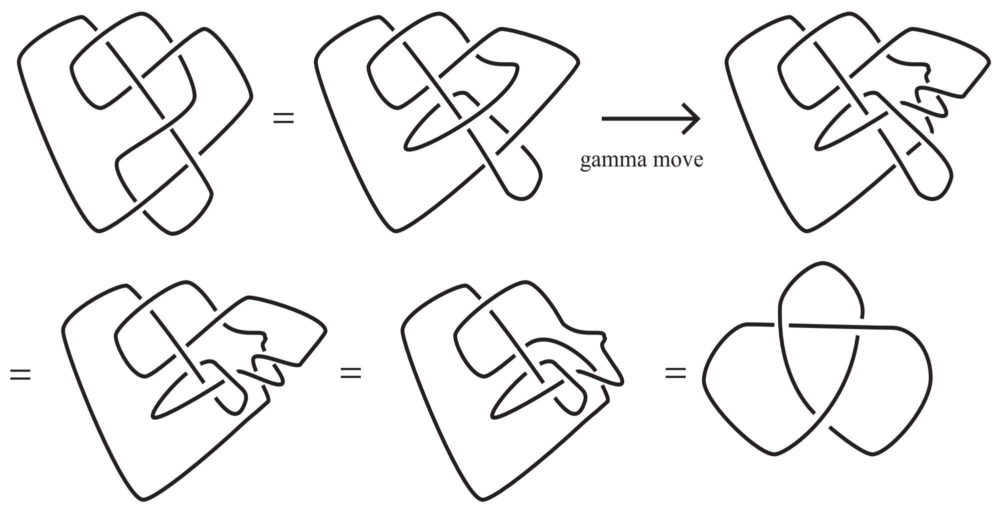

Figure 9

It is easily seen that each knot K of $\{4_1, 5_2, 6_1, 6_3\}$ is non-trivial and not isotopic to the trefoil knot. Thus we have $\Gamma(K) \geq 1$. This completes the proof.

Proof of Theorem 1.2. We can show that $u(K_n) = 2n$ by using the Nakanishi index of a knot [3]. In fact, Nakanishi index is a lower bound of the unknotting number and it is easily seen that the Nakanishi index of K_n is equal to $2n$. (See [4].) Thus we know that $u(K_n) \geq 2n$. On the one hand, K_n can be changed into the unknot by applying $2n$ crossing changes. So we know that $u(K_n) = 2n$. Next, we know that $\Gamma(K_n) \leq n$ and $Arf(K_n) = 0$ as shown in Figure 10. Thus by Proposition 2.3, we know that $2n \leq 3\Gamma(K_n) \leq 3n$.

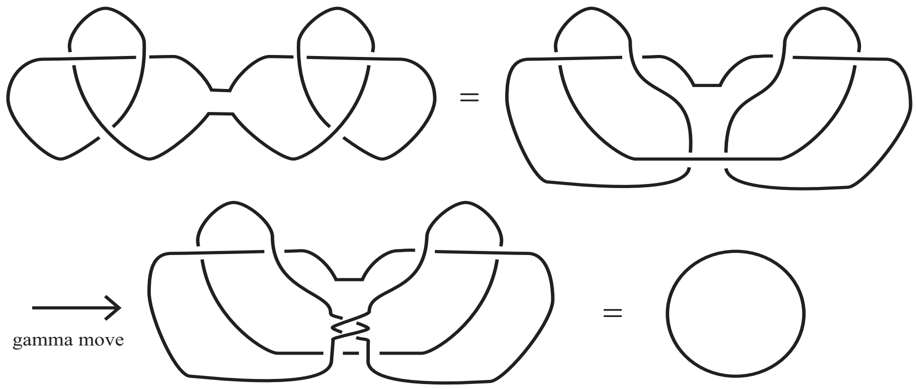

Figure 10

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