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Signed Gordian distances, the Jones polynomial and Rasmussen invariant of knots

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ABSTRACT

We define signed Gordian distances and give criteria for the Gordian distance for knots by using the Jones polynomial, the Q-polynomial and the Rasmussen invariant of a knot. As a result, we have new values of the Gordian distance with small crossing number.

Keywords: Knot; Gordian distance; Jones polynomial

1. Introduction

A *link* is a closed oriented 1-manifold embedded smoothly in the 3-sphere $\mathbb{S}^3$. A *knot* is a link with one connected component. A diagram of a link is a generic projection of a link to the 2-sphere in $\mathbb{S}^3$ with signed double points, called *positive* (or *negative*) *crossings* as in Figure 1. Let K and K' be two knots in $\mathbb{S}^3$. The Gordian distance from K to K', denoted by $d_G(K, K')$, is the minimum number of crossing changes needed to transform a diagram of K into that of K', where the minimum is taken over all diagrams of K and K'. A *positive crossing change* (or *negative crossing change*) of a crossing is changing a positive (or a negative) crossing of a diagram to a

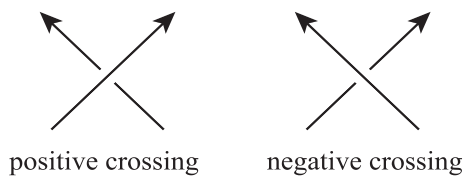


Figure 1

negative (or a positive) crossing. We define $d_{+-}(K, K')$ (or $d_{-+}(K, K')$) as the minimum number of positive (or negative) crossing changes needed to transform a diagram of K into that of K' by $d_G(K, K')$ crossing changes, where the minimum is taken over all diagrams of K and K'. (See [15] in the case when K' is a trivial knot.)

The Jones polynomial V is a Laurent polynomial in one variable t of a link can be defined by the following relation.

(1) $V(\bigcirc; t) = 1$;

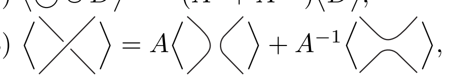
(2) $t^{-1}V(L_+; t) - tV(L_-; t) = -(t^{-1/2} - t^{1/2})V(L_0; t)$.

Here L_+, L_- and L_0 are three links with diagrams differing only near a crossing as in Figure 2.

The Jones polynomial can be calculated from the *Kauffman bracket* $\langle\rangle$ [6]. Let D be an unoriented diagram of a link. Then the Laurent polynomial in one variable A is defined by the following relation:

(1) $\langle \bigcirc \rangle = 1$,

(2) $\langle \bigcirc \cup D \rangle = -(A^2 + A^{-2})\langle D \rangle$,

(3) 

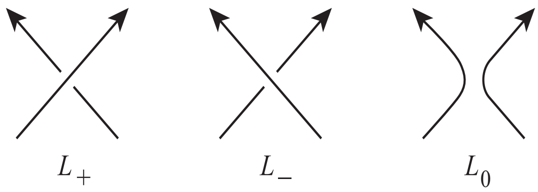


Figure 2

Then the Jones polynomial is obtained as follows:
$V(L;t) = (-t^{-3/4})^{-w(D)} < D > |_{A=t^{-1/4}}$, where $w(D)$ is the writhe of D.

Let $\omega = e^{\pi\sqrt{-1}/3}$ and $\delta = \dfrac{\sqrt{5}-1}{2}$. We need the following theorem.

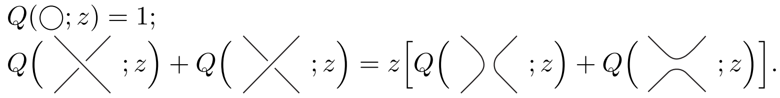
Theorem 1.1. ([7]) *For any link K, we have $V(K;\omega) = \pm(\sqrt{-3})^d\ i^{c(K)-1}$ for some non-negative integer d, where $c(K)$ denotes the number of components of K.*

In this paper, we show the following.

Theorem 1.2. *Let K and K' be two knots in $\mathbb{S}^3$. Suppose that K' is obtained from K by a positive crossing change.*
Let $f(t) = \dfrac{tV(K';t) - V(K;t)}{t-1}$. Then $f(\omega) = \pm\omega^{f'(1)}(\sqrt{-3})^d$ for some non-negative integer d.

Let K and K' two be knots in $\mathbb{S}^3$. By Theorem 1.1, we may assume that $V(K;\omega) = (-1)^{t_1}(\sqrt{-3})^{d_1}$ and $V(K';\omega) = (-1)^{t_2}(\sqrt{-3})^{d_2}$ for some non-negative integers t_1, t_2, d_1 and d_2. Then we have the following theorem which generalizes a result of Traczyk [15].

Theorem 1.3. *If $d_G(K,K') = d_1 - d_2 \geq 1$, then $d_{-+}(K,K') \equiv t_1 - t_2 \mod 2$.*

The Q polynomial $Q(K;z)$ of a link K [1][4] is a Laurent polynomial in one variable z can be defined by the following.
(1) $Q(\bigcirc;z) = 1$;
(2) 

The folloing theorem is a generalization of a result of Stoimenow [13].

Theorem 1.4. *Let K and K' be knots in $\mathbb{S}^3$. If $Q(K;\delta)/Q(K';\delta) = -(-\sqrt{5})^k$, then $d_G(K,K') > |k|$.*

Two links are *concordant* if there is a smooth embedding

$$(nS^1) \times [0,1] \to S^3 \times [0,1]$$

which restricts to the given links

$$(nS^1) \times \{i\} \to S^3 \times \{i\}$$

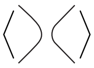
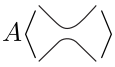
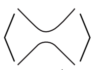
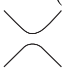
where $i = 0, 1$. The set of concordance classes of knots forms an abelian group under connected sum. The group is called the *knot concordance group*.
Rasmussen has defined a concordance invariant $s(K)$ of a knot K from Lee's cohomology [10]. We call the invariant the *Rasmussen invariant*. Main properties of Rasmussen invariant are summarized as follows.

Theorem 1.5. *Let K, K_1 and K_2 be three knots in S^3. Then we have the following.*
(1) *The Rasmussen invariant s induces a homomorphism from the knot concordance group to $\mathbb{Z}$;*
(2) $|s(K)| \leq 2g_4(K)$, *where $g_4(K)$ is the slice genus of K;*
(3) *If K is alternating, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical knot signature of K;*
(4) *If K_2 is obtained from K_1 by performing a single positive crossing change, then $s(K_1) - s(K_2) \in \{0, 2\}$.*

See [9] for $\sigma(K)$. We have the following by making use of this theorem.

Theorem 1.6. *Let K and K' be two knots in S^3. Let $s(K, K') = \dfrac{s(K) - s(K')}{2}$. Then $d_{+-}(K, K') \geq s(K, K')$ and $d_{-+}(K, K') \geq -s(K, K')$. In particular, $d_G(K, K') \geq |s(K, K')|$.*

2. Proofs

Proof of Theorem 1.2. By a hypothesis, we may consider that a diagram D' of K' is obtained from a diagram D of K by a single positive crossing change. We may assume D to have zero writhe adding kinks if necessarily. Note that
$V(K; A^{-4}) = <D>$ and $A^{-6}V(K; A^{-4}) = <D'>$ since $w(D') + 2 = w(D) = 0$. By using the Kauffman bracket relation, we have
(1) A  $+ A^{-1}$  $= V(K; A^{-4})$,
(2) A^{-1}  $+ A$  $= A^{-6}V(K'; A^{-4})$.
Thus $(A^2 - A^{-2})$  $= A^{-5}V(K'; A^{-4}) - A^{-1}V(K; A^{-4})$.
Then we obtain  $= \dfrac{A^{-3}[A^{-2}V(K'; A^{-4}) - A^2V(K; A^{-4})]}{A^2 - A^{-2}} =$
$-A^{-3}\Big[\dfrac{A^{-4}V(K'; A^{-4}) - V(K; A^{-4})}{A^{-4} - 1}\Big] = -A^{-3}\Big[\dfrac{tV(K'; t) - V(K; t)}{t - 1}\Big]$.
Let $f(t) = \dfrac{tV(K'; t) - V(K; t)}{t - 1}$. Note that $f(t) = \dfrac{t(V(K'; t) - 1)}{t - 1} - \dfrac{V(K; t) - 1}{t - 1} + 1$. By a result in ([5], §12), $V(K; t) - 1$, $V(K'; t) - 1$ and $V(\tilde{K}; t) - 1$ have $(t - 1)(t^3 - 1)$ as factors. Then we know that there exists a knot $\tilde{K}$ with the diagram  such that $V(\tilde{K}; t) = t^n f(t)$ for some integer n. Here we have $V'(\tilde{K}; t) = nt^{n-1}f(t) + t^n f'(t)$. By substituting 1, we have $V'(\tilde{K}; 1) = nf(1) + f'(1)$. Thus $f(1) = 1$ and $V'(\tilde{K}; 1) = 0$. Therefore $n = -f'(1)$, and hence we have $V(\tilde{K}, t) = t^{-f'(1)}f(t)$. By Theorem 1.1, we know that $V(\tilde{K}; \omega) = \omega^{-f'(1)}f(\omega) = \pm(\sqrt{-3})^d$ for some non-negative integer d.

Proof of Theorem 1.3. By an assumption that $d_G(K, K') = d_1 - d_2$, we have a sequence of $d_1 - d_2 + 1$ knots $K = K_{d_1-d_2}, K_{d_1-d_2-1}, \ldots, K_1, K_0 = K'$ such that $K = K_{d_1-d_2}$, $K_0 = K'$ and $d_G(K_j, K_{j-1}) = 1$ $(1 \leq j \leq d_1 - d_2)$. By Theorem 1.1, we may assume that $V(K_i; \omega) = (-1)^{s_i}(\sqrt{-3})^{\ell_i}$ for some integers s_i and ℓ_i $(0 \leq i \leq d_1 - d_2)$, where $\ell_{d_1-d_2} = d_1$, $\ell_0 = d_2$, $s_{d_1-d_2} = t_1$ and $s_0 = t_2$. First we show that $\ell_j - \ell_{j-1} = 1$ $(1 \leq j \leq d_1 - d_2)$ as follows.

Let $\ell_j - \ell_{j-1} = n_j$ $(1 \leq j \leq d_1 - d_2)$ and suppose that $n_j \geq 2$ for some j. In the case when K_{j-1} is obtained from K_j by a positive crossing change, by substituting ω for t in the second relation of the definition of the Jones polynomial, we have

$\omega^{-1}V(K_j;\omega)-\omega V(K_{j-1};\omega)=e^{-\frac{\pi i}{3}}((-1)^{s_j}(\sqrt{-3})^{\ell_{j-1}+n_j})-e^{\frac{\pi i}{3}}((-1)^{s_{j-1}}(\sqrt{-3})^{\ell_{j-1}})$
$=(\sqrt{3}i)^{\ell_{j-1}}((-1)^{s_j}(\sqrt{3}i)^{n_j}\frac{1-\sqrt{3}i}{2}-(-1)^{s_{j-1}}\frac{1+\sqrt{3}i}{2})$
$=(\sqrt{3}i)^{\ell_{j-1}}\left\{\frac{(-1)^{s_j}(\sqrt{3}i)^{n_j}-(-1)^{s_j}(\sqrt{3}i)^{n_j+1}-(-1)^{s_{j-1}}-(-1)^{s_{j-1}}\sqrt{3}i}{2}\right\}.$

Let $P_{n_j}=\frac{(-1)^{s_j}(\sqrt{3}i)^{n_j}-(-1)^{s_j}(\sqrt{3}i)^{n_j+1}-(-1)^{s_{j-1}}-(-1)^{s_{j-1}}\sqrt{3}i}{2}$. If $n_j=2m_j(\geq 2)$ for a positive integer m_j, then
$P_{n_j}=\frac{(-1)^{s_j}(-3)^{m_j}-(-1)^{s_j}(-3)^{m_j}(\sqrt{3}i)-(-1)^{s_{j-1}}-(-1)^{s_{j-1}}\sqrt{3}i}{2}$
$=\frac{(-1)^{s_j}(-3)^{m_j}-(-1)^{s_{j-1}}}{2}-\frac{(-1)^{s_j}(-3)^{m_j}+(-1)^{s_{j-1}}}{2}\sqrt{3}i.$

In this case, $|P_{n_j}|^2=(-3)^{m_j}((-3)^{m_j}+(-1)^{s_j+s_{j-1}})+1$. If $n_j=2m_j+1(\geq 3)$ for a positive integer m_j, then
$P_{n_j}=\frac{(-1)^{s_j}(-3)^{m_j}(\sqrt{3}i)-(-1)^{s_j}(-3)^{m_j+1}-(-1)^{s_{j-1}}-(-1)^{s_{j-1}}\sqrt{3}i}{2}$
$=-\frac{(-1)^{s_j}(-3)^{m_j+1}+(-1)^{s_{j-1}}}{2}+\frac{(-1)^{s_j}(-3)^{m_j}-(-1)^{s_{j-1}}}{2}\sqrt{3}i.$
In this case, $|P_{n_j}|^2=(-3)^{m_j+1}(-(-3)^{m_j}+(-1)^{s_j+s_{j-1}})+1$.
In both cases, $|P_{n_j}|^2$ cannot be equal to 3^d for any non-negative integer d. Then we know that $\omega^{-1}V(K_j;\omega)-\omega V(K_{j-1};\omega)$ cannot have a form $\pm(\sqrt{-3})^d$ for any non-negative integer d. This contradicts to Theorem 1.1.
In the case when K_{j-1} is obtained from K_j by a negative crossing change, by using the same argument, we know that $\omega^{-1}V(K_{j-1};\omega)-\omega V(K_j;\omega)$ also cannot have the form $\pm(\sqrt{-3})^d$ for any non-negative integer d if $n_j\geq 2$. Thus we know that $n_j\leq 1$ for any j $(1\leq j\leq d_1-d_2)$. Then since $\sum_{p=1}^{d_1-d_2}n_p=\sum_{p=1}^{d_1-d_2}(\ell_p-\ell_{p-1})=d_1-d_2$, $1\geq n_k=d_1-d_2-\sum_{p=1,p\neq k}^{d_1-d_2}n_p\geq 1$ for any integer k $(1\leq k\leq d_1-d_2)$. Therefore $n_k=1$ for any integer k $(1\leq k\leq d_1-d_2)$.
In the case when $n_j=1$ $(1\leq j\leq d_1-d_2)$, we know that $P=\frac{3(-1)^{s_j}-(-1)^{s_{j-1}}}{2}+\frac{(-1)^{s_j}-(-1)^{s_{j-1}}}{2}\sqrt{3}i$. Then $|P|^2=4-3(-1)^{s_j+s_{j-1}}=4-3(-1)^{s_j-s_{j-1}}=7$ if $s_j-s_{j-1}\equiv 1 \mod 2$. Thus $\omega^{-1}V(K_j;\omega)-\omega V(K_{j-1};\omega)$ does not have the form $\pm(\sqrt{-3})^d i$ for any non-negative integer d if $s_j-s_{j-1}\equiv 1 \mod 2$ by the above calculation. We also know that $\omega^{-1}V(K_{j-1};\omega)-\omega V(K_j;\omega)$ does not have a form $\pm(\sqrt{-3})^d$ for any non-negative integer d if $s_j-s_{j-1}\equiv 0 \mod 2$ by using the same argument. (In this case, $|P|^2=4+3(-1)^{s_j-s_{j-1}}$.)
Therefore, we know that if K_{j-1} is obtained from K_j by a positive crossing change then s_j-s_{j-1} is even (i.e. $(-1)^{s_j}=(-1)^{s_{j-1}}$), and if K_{i-1} is obtained from K_i by a negative crossing change, then s_j-s_{j-1} is odd (i.e. $(-1)^{s_j}=-(-1)^{s_{j-1}}$). Now we have $(-1)^{s_1}=(-1)^r(-1)^{s_2}$, where r is the number of the negative crossing changes in the sequence of crossing changes of knots $K_{d_1-d_2},\ldots,K_1$. Then $(-1)^{r-(s_1-s_2)}=1$, and hence $r\equiv s_1-s_2 \mod 2$.

Proof of Theorem 1.4. We show the theorem by an induction with respect to the Gordian distance. Suppose that $d_G(K,K')=1$. By an argument in the proof of Theorem 4.1 [13], $Q(K,\delta)/Q(K',\delta)\in\{\pm 1,-(\sqrt{5})^{\pm 1}\}$ if K' is obtained from K by a single crossing change. Thus if $Q(K,\delta)/Q(K',\delta)=-(-\sqrt{5})^k$, then $|k|=0$. Therefore $d_G(K,K')=1>0=|k|$. Now we assume that the result holds in the case when $d_G(K,K')=m-1$ for a positive integer $m\geq 2$. Suppose that $d_G(K,K')=m\geq 1$ and $Q(K,\delta)/Q(K',\delta)=-(-\sqrt{5})^{k'}$ for some integer k'. Then there exist $m+1$ knots $K_0, K_1, \dots, K_m$ such that $K = K_0$, $K' = K_m$ and $d_G(K_i, K_{i+1}) = 1$ $(0 \le i \le m-1)$. Note that $Q(K,\delta)/Q(K',\delta) = Q(K_0,\delta)/Q(K_{m-1},\delta) \times Q(K_{m-1},\delta)/Q(K_m,\delta) = -(-\sqrt{5})^{k'}$. If $Q(K_0,\delta)/Q(K_{m-1},\delta) = -(-\sqrt{5})^{\overline{k}}$ for some integer $\overline{k}$, then we have $|\overline{k}| < m-1$ and $Q(K_{m-1},\delta)/Q(K_m,\delta) = -(-\sqrt{5})^{k'}/-(-\sqrt{5})^{\overline{k}} = (-\sqrt{5})^{k'-\overline{k}} \in \{\pm 1, -(\sqrt{5})^{\pm 1}\}$ since $d_G(K_0, K_{m-1}) = m-1$ and $d_G(K_{m-1}, K_m) = 1$. Then we have $|k'-\overline{k}| \le 1$, and hence $|k'| \le |k'-\overline{k}|+|\overline{k}| < m = d_G(K_0, K_m)$. If $Q(K_0,\delta)/Q(K_{m-1},\delta) \ne -(-\sqrt{5})^{\overline{k}}$ for any integer $\overline{k}$, that is, $Q(K_0,\delta)/Q(K_{m-1},\delta) = (-\sqrt{5})^{\tilde{k}}$ for some integer $\tilde{k}$, then $d_G(K_0, K_{m-1}) \ge |\tilde{k}|$ and $Q(K_{m-1},\delta)/Q(K_m,\delta) = -(-\sqrt{5})^{k'}/(-\sqrt{5})^{\tilde{k}} = -(-\sqrt{5})^{k'-\tilde{k}} \in \{\pm 1, -(\sqrt{5})^{\pm 1}\}$. Thus we have $k' - \tilde{k} = 0$. Therefore $|k'| \le |k'-\tilde{k}| + |\tilde{k}| \le d_G(K_0, K_{m-1}) = m-1 < m = d_G(K_0, K_m)$. This completes the proof.

Proof of Theorem 1.6. If $s(K,K') \ge 0$, then we need to perform at least $s(K,K')$ positive crossing changes to obtain K' from K by Theorem 1.5(4). Thus we have $d_{+-}(K,K') \ge s(K,K')$. If $s(K,K') \le 0$, then, by using the same idea, we have $d_{+-}(K,K') = d_{-+}(K',K) \ge s(K',K) = -s(K,K')$.

3. Examples

For a knot K, K^* denotes the mirror image of K. In this section, we detect $d_G(3_1\sharp 4_1, 5_1)$, $d_G(3_1\sharp 3_1, 4_1)$, $d_G(3_1^*\sharp 3_1^*, 4_1)$ and $d_G(3_1\sharp 3_1^*, 7_3)$ which are unknown values in a table of Darcy and Sumners in [3] by using Theorems 1.2, 1.3, 1.4 and 1.6. We also detect $d_G(4_1\sharp 4_1, 3_1)$, $d_G(4_1\sharp 4_1, 3_1^*)$, $d_G(4_1\sharp 4_1, 5_2)$, $d_G(4_1\sharp 4_1, 5_2^*)$ and $d_G(4_1\sharp 4_1, 6_3)$ which are unknown values in a table [2] of Darcy by using Theorem 1.4. Moreover we determine some other Gordian distances, by using Theorem 1.6, which cannot be obtained from Theorems 1.2, 1.3 and 1.4. (See Figures 3, 4, 5 ,6 and 7 for a diagram of a knot A_b.)

Example 3.1.
(1) $d_G(3_1\sharp 4_1, 5_1) = 2$.
(2) $d_{+-}(3_1\sharp 4_1, 5_1) = 0$ and $d_{-+}(3_1\sharp 4_1, 5_1) = 1$.
We prove (1) by using Theorems 1.2 and 1.5 as follows. It is easily see that $d_G(3_1\sharp 4_1, 5_1) \le 2$ since $d_G(0_1, 4_1) = 1$ and $d_G(3_1, 5_1) = 1$. Suppose that $3_1\sharp 4_1$ is obtained from 5_1 by a positive crossing change. Then
$$\frac{\omega V(3_1\sharp 4_1;\omega) - V(5_1;\omega)}{\omega - 1} = \frac{\omega(-\sqrt{3}i) - (-1)}{\omega - 1} = \frac{-1-\omega(-\sqrt{3}i)}{1-\omega} = \frac{-1+\sqrt{3}i\omega}{\overline{\omega}} = \omega(\sqrt{3}i\omega - 1) = -2-\sqrt{3}i.$$
This contradicts to Theorem 1.2. Suppose that 5_1 is obtained from $3_1\sharp 4_1$ by a positive crossing change. Then by Theorem 1.5(4), we know that $s(3_1\sharp 4_1) - s(5_1) \in \{0, 2\}$. On the other hand, $s(3_1\sharp 4_1) = \sigma(3_1\sharp 4_1) = 2$ and $s(5_1) = \sigma(5_1) = 4$ by Theorem 1.5(3). This is a contradiction. Therefore we have $d_G(3_1\sharp 4_1, 5_1) \ge 2$. In fact, $d_G(3_1\sharp 4_1, 5_1) \le 2$ as shown in Figure 3, and hence $d_G(3_1\sharp 4_1, 5_1) = 2$. We cannot use Theorem 1.4 to show this because $Q(3_1\sharp 4_1, \delta) = Q(5_1, \delta) = \sqrt{5}$. We cannot also use Theorem 1.6 since $|s(3_1\sharp 4_1, 5_1)| = |-1| = 1$. Next we prove (2) by Theorem 1.6. We know that $d_{+-}(3_1\sharp 4_1, 5_1) = 0$ and $d_{-+}(3_1\sharp 4_1, 5_1) \le 1$ as shown in Figure 3. By Theorem 1.6, $d_{-+}(3_1\sharp 4_1, 5_1) \ge -s(3_1\sharp 4_1, 5_1) = 1$. Thus we have $d_{-+}(3_1\sharp 4_1, 5_1) = 1$.
We can also prove (1) by using Theorems 1.3 and 1.6 as follows. Since $V(3_1\sharp 4_1;\omega) = -\sqrt{-3}$ and $V(5_1;\omega) = -1$, if $d_G(3_1\sharp 4_1, 5_1) = 1$, then $d_{-+}(3_1\sharp 4_1, 5_1) \equiv 0 \bmod 2$ by Theorems 1.3, and hence $d_{-+}(3_1\sharp 4_1, 5_1) = 0$. However, as shown above, $d_{-+}(3_1\sharp 4_1, 5_1) = 1$ by Theorem 1.6. This is a contradiction.

Example 3.2.
(1) $d_G(3_1\sharp 3_1, 4_1) = d_G(3_1^*\sharp 3_1^*, 4_1) = 3$.
(2) $d_{+-}(3_1\sharp 3_1, 4_1) = d_{-+}(3_1^*\sharp 3_1^*, 4_1) = 2$.

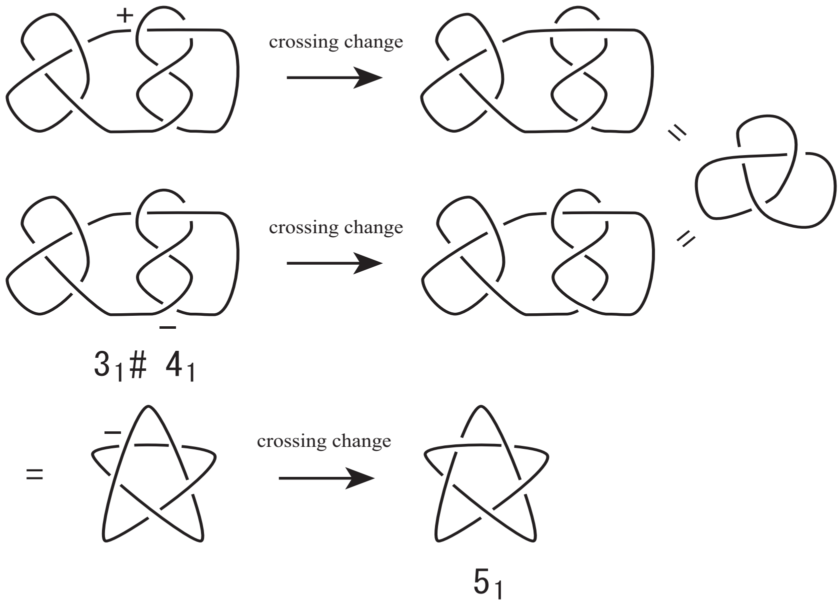


FIGURE 3

(3) $d_{-+}(3_1\sharp 3_1, 4_1) = d_{+-}(3_1^*\sharp 3_1^*, 4_1) = 0$.
By Theorem 1.6, $d_{-+}(3_1^*\sharp 3_1^*, 4_1) \geq -s(3_1^*\sharp 3_1^*, 4_1) = -\dfrac{s(3_1^*\sharp 3_1^*) - s(4_1)}{2} = 2$ since $s(3_1^*\sharp 3_1^*) = -4$ and $s(4_1) = 0$. Suppose that $d_G(3_1^*\sharp 3_1^*, 4_1) = 2$. By Theorem 1.3, $d_{-+}(3_1^*\sharp 3_1^*, 4_1) \equiv 1 \bmod 2$ since $V(3_1^*\sharp 3_1^*; \omega) = (-1)^0(\sqrt{-3})^2$ and $V(4_1; \omega) = (-1)^1(\sqrt{-3})^0$. Then we have $d_{-+}(3_1^*\sharp 3_1^*, 4_1) = 1$. This is a contradiction. Thus we know that $d_G(3_1^*\sharp 3_1^*, 4_1) \geq 3$. As shown in Figure 4, we know that $d_G(3_1^*\sharp 3_1^*, 4_1) \leq 3$, $d_{-+}(3_1^*\sharp 3_1^*, 4_1) \leq 2$ and $d_{+-}(3_1^*\sharp 3_1^*, 4_1) = 0$, and hence $d_G(3_1^*\sharp 3_1^*, 4_1) = 3$, $d_{-+}(3_1^*\sharp 3_1^*, 4_1) = 2$. Since $d_G(K, K') = d_G(K^*, (K')^*)$, $d_{-+}(K, K') = d_{+-}(K^*, (K')^*)$ and $d_{+-}(K, K') = d_{-+}(K^*, (K')^*)$ for any two knots K and K', we have $d_G(3_1\sharp 3_1, 4_1) = 3$, $d_{+-}(3_1\sharp 3_1, 4_1) = 2$ and $d_{-+}(3_1\sharp 3_1, 4_1) = 0$.

Example 3.3.
(1) $d_G(3_1\sharp 3_1^*, 7_3) = 3$.
(2) $d_{+-}(3_1\sharp 3_1^*, 7_3) = 0$.
We can obtain the above values by using the same argument as in Example 3.2. In fact, $d_{-+}(3_1\sharp 3_1^*, 7_3) \geq 2$ by Theorem 1.6. On the other hand, we have $d_{-+}(3_1\sharp 3_1^*, 7_3) = 1$ by Theorem 1.3 since $V(3_1\sharp 3_1^*; \omega) = (-1)(\sqrt{-3})^2$ and $V(7_3; \omega) = (-1)^0(\sqrt{-3})^0$ if we assume that $d_G(3_1\sharp 3_1^*, 7_3) = 2$. Thus $d_G(3_1\sharp 3_1^*, 7_3) \geq 3$ and we obtain $d_G(3_1\sharp 3_1^*, 7_3) \leq 3$ and $d_{+-}(3_1\sharp 3_1^*, 7_3) = 0$ as shown in Figure 5, and hence $d_G(3_1\sharp 3_1^*, 7_3) = 3$. However we cannot detect $d_{-+}(3_1\sharp 3_1^*, 7_3)$. (This value is larger than or equal to 2 by the above argument.)

Remark. Miyazawa has given a criterion of the Gordian distance of knots by using the HOMFLY polynomial in [8]. The value $d_G(3_1\sharp 4_1, 5_1)$ is also obtained by making use of the result. On the other hand, $d_G(3_1\sharp 3_1, 4_1)$, $d_G(3_1^*\sharp 3_1^*, 4_1)$ and $d_G(3_1\sharp 3_1^*, 7_3)$ cannot be obtained from it.

Example 3.4.
(1) $d_G(4_1\sharp 4_1, 3_1) = d_G(4_1\sharp 4_1, 3_1^*) = 3$;

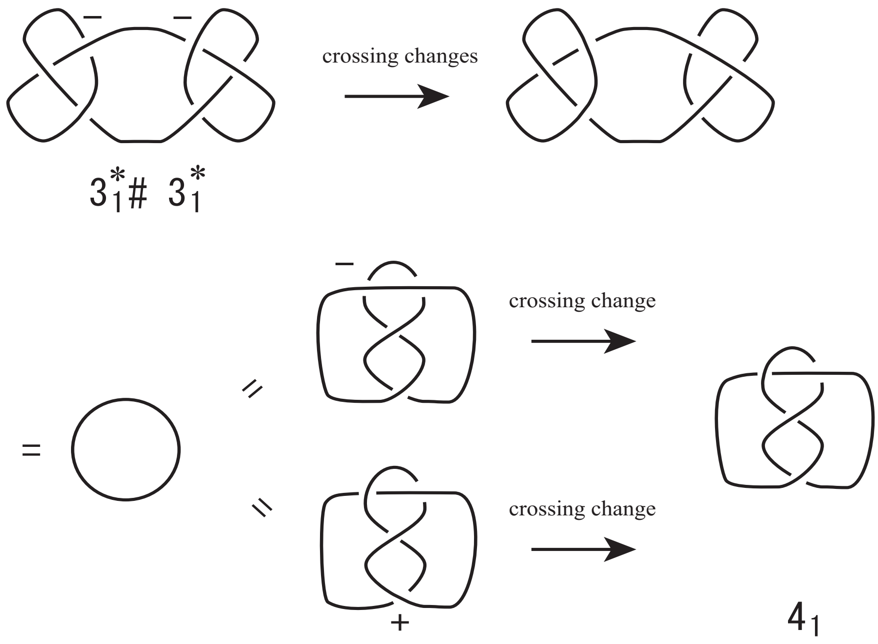


FIGURE 4

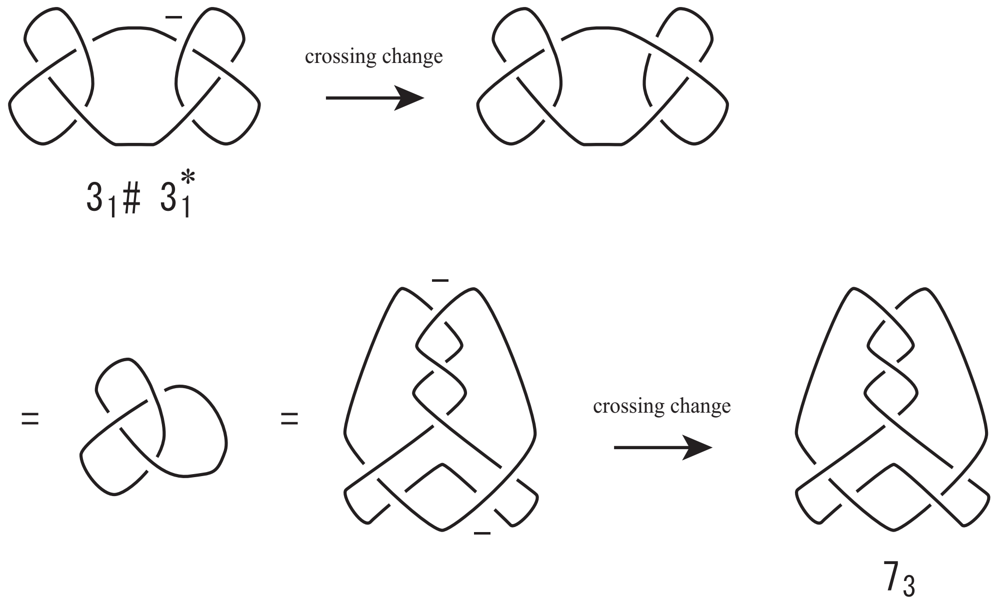


FIGURE 5

(2) $d_G(4_1\sharp 4_1, 5_2) = d_G(4_1\sharp 4_1, 5_2^*) = 3$;
(3) $d_G(4_1\sharp 4_1, 6_3) = 3$.
We know that $d_G(4_1\sharp 4_1, 3_1)$, $d_G(4_1\sharp 4_1, 3_1^*)$, $d_G(4_1\sharp 4_1, 5_2)$, $d_G(4_1\sharp 4_1, 5_2^*)$ and $d_G(4_1\sharp 4_1, 6_3)$ are less than or equal

to 3 since $d_G(3_1, 0_1) = d_G(4_1, 0_1) = d_G(5_2, 0_1) = d_G(6_3, 0_1) = 1$. Then we know that the equations (1), (2) and (3) are obtained from Theorem 1.4 since $Q(3_1; \delta) = Q(5_2; \delta) = Q(6_3; \delta) = -1$ and $Q(4_1; \delta) = -\sqrt{5}$. These numbers are undecided in the table of I. Darcy [2]. We cannot detect them by using Theorem 1.2, 1.3 bacause of the values of the Jones polynomials and we cannot also use Theorems 1.6 to detect them since $s(4_1\sharp 4_1) = \sigma(4_1\sharp 4_1) = 0$, $s(3_1)$(or $|s(3_1^*)|$) $= \sigma(3_1)$(or $|\sigma(3_1^*)|$) $= 2$, $s(5_2)$(or $|s(5_2^*)|$) $= \sigma(5_2)$(or $|\sigma(5_2^*)|$) $= 2$ and $s(6_1) = \sigma(6_1) = 0$.

Example 3.5. Let Y be a knot in the set $\{3_1, 5_2, 6_2, 7_2, 7_6, 8_7, 8_{11}, 8_{14}\}$ (see Figure 7). Let $X = Y^*$. Then
(1) $d_G(X, 10^*_{145}) = d_{-+}(X, 10^*_{145}) = 3$, $d_{+-}(X, 10^*_{145}) = 0$,
(2) $d_G(X, 10_{154}) = d_{-+}(X, 10_{154}) = 4$, $d_{+-}(X, 10_{154}) = 0$,
(3) $d_G(X, 10^*_{161}) = d_{-+}(X, 10^*_{161}) = 4$, $d_{+-}(X, 10^*_{161}) = 0$.
It is easily see that $d_G(X, 0_1) \leq 1$ and $d_G(10^*_{145}, 0_1) \leq 2$, $d_G(10_{154}, 0_1) \leq 3$ and $d_G(10^*_{161}, 0_1) \leq 3$. (If we change signed crossings in Figures 6 and 7, we obtain tirivial knots.) Then we know that $d_G(X, 10^*_{145}) \leq 3$, $d_G(X, 10_{154}) \leq 4$ and $d_G(X, 10^*_{161}) \leq 4$. Now we show that $d_{-+}(X, 10^*_{145}) \geq 3$, $d_{-+}(X, 10_{154}) \geq 4$ and $d_{-+}(X, 10^*_{161}) \geq 4$. We can show that 10^*_{145}, 10_{154} and 10^*_{161} as in Figure 3 are quasipositive knots [11][12][14]. In fact, 10_{145} and 10^*_{161} are the closure of quasipositive braids
$\sigma_2^2\sigma_3\sigma_1(\sigma_2^{-1}\sigma_1\sigma_2)(\sigma_3^{-1}\sigma_2\sigma_3)\sigma_1$ and $(\sigma_1\sigma_2\sigma_1^{-1})\sigma_2\sigma_1^2\sigma_2^3\sigma_1$
respectively and 10_{154} is a positive knot with a positive diagram consists of 10 crossings and 5 Seifert circles as described in Figure 6.

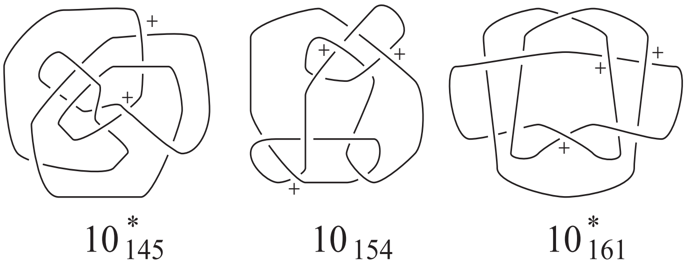


Figure 6

Then by using a result of Shumakovitch [14], we know that $s(10^*_{145}) = 4$ and $s(10_{154}) = s(10^*_{161}) = 6$. On the other hand, $s(X) = -s(Y) = -\sigma(X) = -2$ since each X is an alternating knot. (For each knot Y, by a crossing change of the positive crossing signed in Figure 7, we have a trivial knot. Thus by Theorem 1.5(4), we know that $s(Y) \geq 0$.) Thus by using Theorem 1.6, we have $d_{-+}(X, 10^*_{145}) \geq -\dfrac{s(X) - s(10^*_{145})}{2} = 3$, $d_{-+}(X, 10_{154}) \geq -\dfrac{s(X) - s(10_{154})}{2} = 4$ and $d_{-+}(X, 10^*_{161}) \geq -\dfrac{s(X) - s(10^*_{161})}{2} = 4$. Thus we have $d_{-+}(X, 10^*_{145}) = d_G(X, 10^*_{145}) = 3$, $d_{-+}(X, 10_{154}) = d_G(X, 10_{154}) = 4$ and $d_{-+}(X, 10^*_{161}) = d_G(X, 10^*_{161}) = 4$, and hence $d_{+-}(X, 10^*_{145}) = d_{+-}(X, 10_{154}) = d_{+-}(X, 10^*_{161}) = 0$. However we cannot use Theorem 1.4 to detect them because $Q(10^*_{145}; \delta) = Q(10_{154}; \delta) = -1$ and $Q(10^*_{161}; \delta) = \sqrt{5}$. (See Figure 8 for a value of $Q(X; \delta)$.)

Here we give the following problem.

Problem. For any four knots K_0, K_1, K_2, K_3, does the next equality hold?

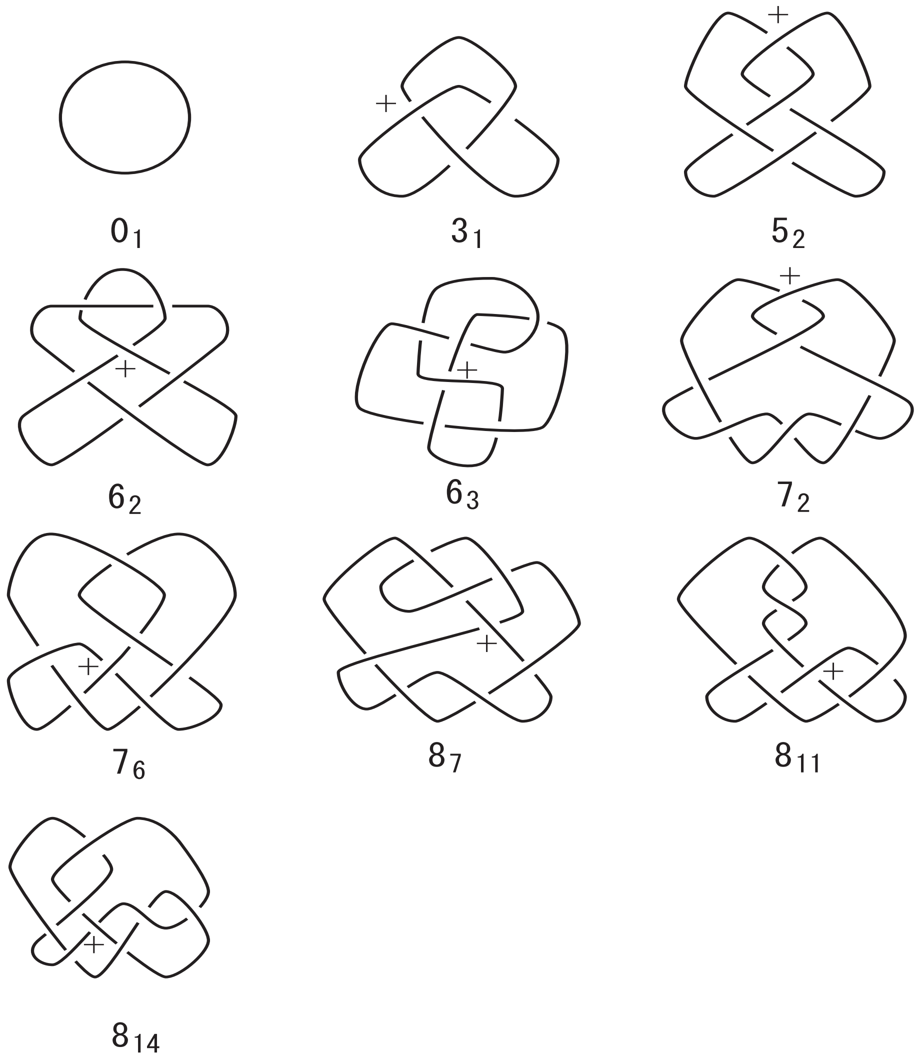


FIGURE 7

$$d_G(K_0\sharp K_1, K_2\sharp K_3) = \min\{d_G(K_0, K_2) + d_G(K_1, K_3),\ d_G(K_0, K_3) + d_G(K_1, K_2)\}.$$

In the first example, $d_G(3_1, 0_1) = d_G(3_1, 5_1) = d_G(4_1, 0_1) = 1$. Thus we have
$d_G(3_1\sharp 4_1, 0_1\sharp 5_1) = \min\{d_G(3_1, 0_1) + d_G(4_1, 5_1),\ d_G(3_1, 5_1) + d_G(4_1, 0_1)\} = 2$
even if we cannot detect $d_G(4_1, 5_1)$. (In fact, we know that $d_G(4_1, 5_1) \geq 2$ by making use of Theorem 3.1.)

We list signatures, special values of the Jones polynomial and the Q polynomial for knots with up to 8 crossings (Figure 8.) (Here we set $a = \sqrt{-3}$ and $b = \sqrt{5}$.)

K	σ	$V(K;\omega)$	$Q(K;\delta)$	K	σ	$V(K;\omega)$	$Q(K;\delta)$
3_1	2	a	-1	8_5	4	a	1
4_1	0	-1	$-b$	8_6	2	1	-1
5_1	4	-1	b	8_7	2	1	-1
5_2	2	-1	-1	8_8	2	1	b
6_1	0	$-a$	1	8_9	0	1	$-b$
6_2	2	1	1	8_{10}	2	a	-1
6_3	0	1	-1	8_{11}	2	a	-1
7_1	-6	-1	-1	8_{12}	0	-1	1
7_2	2	1	1	8_{13}	0	-1	1
7_3	4	1	-1	8_{14}	2	-1	1
7_4	2	$-a$	b	8_{15}	-4	a	-1
7_5	-4	-1	-1	8_{16}	2	1	b
7_6	2	-1	1	8_{17}	0	1	-1
7_7	0	$-a$	1	8_{18}	0	3	b
8_1	0	1	-1	8_{19}	6	$-a$	-1
8_2	-4	-1	-1	8_{20}	0	$-a$	1
8_3	0	-1	-1	8_{21}	2	a	$-b$
8_4	2	-1	1				

Figure 8

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