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Positive knots and Rasmussen's invariant

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ABSTRACT

We define the positive (resp. negative) index of a knot as the minimal number of crossing changes which are needed to make the knot into a positive (resp. negative) knot. We give lower bounds for the invariants of knots by using the Rasmussen invariant. We detect the invariants for a torus knot. We also compute the invariant for most knots with crossing number less than 8.
Keywords: positive knot; Rasmussen invariant; unknotting number

1. Introduction

A *link* is smoothly embedded circles in the 3-sphere $\mathbb{S}^3$. A *knot* is a link with one connected component. We assume that every link is oriented. A diagram of a link is a generic projection of a link to the 2-sphere in $\mathbb{S}^3$ with signed double points, called *positive* (or *negative*) *crossings* as in Figure 1.

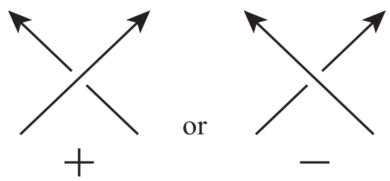


Figure 1

We say that a knot is *positive* if it admits a diagram with all positive crossings. We define an invariant u_P (resp. u_N) of a link as the minimal number of crossing changes which are needed to make the knot into a positive (resp. negative) knot. We call the number the *positive* (*resp. negative*) *index* of a link. Let $u(K)$ be the unknotting number of a knot K. In [4], Rasmussen has introduced an effective concordance invariant $s(K)$ of a knot K which gives a lower bound for the unknotting number. Let $K!$ be the mirror image of a knot K with reversed orientation. In this paper, we show the following.

Theorem 1.1. *Let K be a knot in $\mathbb{S}^3$.*
(1) $-\dfrac{s(K)}{2} \leq u_P(K) \leq u(K)$;
(2) $\dfrac{s(K)}{2} \leq u_N(K) \leq u(K)$;
(3) $u_P(K) = u_N(K!)$.

Corollary 1.2. *Let $T(p,q)$ be the torus knot of type (p,q).*
(1) $u_P(T(p,q)) = 0$;
(2) $u_N(T(p,q)) = \dfrac{(p-1)(q-1)}{2}$.

We say that a knot is *slice* if it bounds a smooth disc in the 4-ball.

Proposition 1.3. *Let K be a knot with unknotting number one. If K is slice, then both of the positive index and the negative index equal one.*

Corollary 1.4. *There exists a knot K such that $s(K) = 0$ and $u_P(K) = u_N(K) = 1$.*

This paper is organized as follows. In Section 2, we will recall properties of Rasmussen invariant and prove our results. In Section 3, we will give a result of computations of the positive and the negative indices for knots with crossing number less than 8.

2. Rasmussen invariant

In a paper of Khovanov published in 2000 [2], a cohomology has been constructed from a diagram of a knot or a link via (1+1)-TQFT. Lee modified Khovanov's TQFT and simplified the cohomology [3], and Rasmussen has defined an effective concordance invariant $s(K)$ of a knot K from Lee's cohomology [4]. (We call the invariant the *Rasmussen invariant*.) Rasmussen has detected the unknotting number of a torus knot by using the invariant and proved Milnor's conjecture combinatorially.
The *slice genus* of a knot K in S^3 is the least integer g such that K is the boundary of a connected, orientable 2-manifold S with genus g smoothly embedded in the 4-ball B^4 bounded by S^3. The slice genus of K is denoted by $g_4(K)$.
Two links are *concordant* if there is a smooth embedding

$$(nS^1) \times [0,1] \to S^3 \times [0,1]$$

which restricts to the given links

$$(nS^1) \times \{i\} \to S^3 \times \{i\}$$

where $i = 0, 1$. The set of concordance classes of knots forms an abelian group under connected sum [1]. The group is called the *knot concordance group*.
Main properties of Rasmussen invariant are summarized as follows [4].

Theorem 2.1. *If K is a knot in S^3, then we have the following.*
(1) *s induces a homomorphism from the knot concordance group to $\mathbb{Z}$;*
(2) *$|s(K)| \leq 2g_4(K)$, where $g_4(K)$ is the slice genus of K;*
(3) *If K is alternating, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical knot signature of K;*
(4) *If K_1 is obtained from K_2 by performing a single crossing change, then $|s(K_1) - s(K_2)| \leq 2$.*

Theorem 2.2. *If K is a positive knot, $s(K) = 2g_4(K) = 2g(K)$, where $g(K)$ is the ordinary genus of K.*

Theorem 2.3. *If K is a knot, then we have $s(K) = -s(K!)$.*

Proof of Theorem 1.1. Since the unknot is positive and negative, we have the right inequalities of (1) and (2), $u_P \leq u(K)$ and $u_N(K) \leq u(K)$. The equation (3) follows from the definition immediately. The left inequality of (2), $s(K)/2 \leq u_N(K)$, follows from that of (1), $-s(K!)/2 \leq u_P(K!)$, the equality (3) and Theorem 2.3. Hence, it is suffice to show that $-s(K)/2 \leq u_P(K)$. If $s(K)$ is positive, then the inequality holds. So we assume that $s(K) \leq 0$ and let $-s(K)/2 = n_K$. (The integer $s(K)$ is always even.) Then we need to perform at least n_K crossing changes to have a knot with non-negative Rasmussen invariant by Theorem 2.1(4). By Theorem 2.2, if K is a positive knot, then $s(K) \geq 0$. Thus by the definition, we have $n_K \leq u_P(K)$.

Proof of Corollary 1.2. The torus knot $T(p, q)$ is a positive knot. Hence $u_P(T(p, q)) = 0$. By a result in [4], it is shown that $s(K) = 2u(K) = (p-1)(q-1)$. Thus, by Theorem 1.1(2), we know that $u_N(T(p, q)) = (p-1)(q-1)/2$.

Proof of Proposition 1.3. Let K be a slice knot with unknotting number one. Then we know that $s(K) = 0$ and K is non-trivial. By Theorem 2.2, if K is non-trivial positive knot, then $s(K)$ is positive. Thus, we need to perform at least one crossing change to make K into a positive knot by Theorem 2.1(4). Therefore, we have $u_P(K) \geq 1$. By Theorem 2.2 and 2.3, we know that non-trivial negative knot have negative Rasmussen invariant. So by using the same method as above, we know that $u_N(K) \geq 1$. Then by Theorem 1.1, we have $u_P(K) = u_N(K) = 1$.

Proof of Corollary 1.4. Let K be a *Stevedore's knot* as shown in Figure 2. Then we know that K is a slice knot and $u(K) = 1$. Thus, Corollay 1.4 follows from Proposition 1.3.

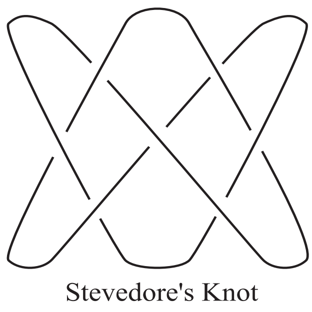


FIGURE 2

3. EXAMPLES

In this section, we calculate the positive index and the negative index for some knots. Finally, we give a list of computations for knots with crossing number less than 8.

We need the following proposition.

Proposition 3.1. *Let K be a positive knot and let $TB(K)$ be the maximal Thurston-Bennequin invariant of a knot K. If K be a positive knot then $TB(K) + 1 = s(K)$.*

Proof. The result follows from results in [6] and [4].

Example 3.2. Let K be the knot 8_2 in Rolfsen's table [5] (See Figure 3). Then $s(K) = \sigma(K) = -4$ and $-s(K)/2 = 2 \leq u_P(K) \leq u(K) = 2$. Thus $u_P(K) = 2$. The knot K is transformed into a negative knot by performing a single crossing changes as shown in Figure 2. (Change the crossing marked by a dotted circle.) Hence we know that $u_N(K) \leq 1$. However, we can show that $TB(K!) \neq s(K!) - 1$. Thus we have $u_N(K) = 1$ by Proposition 3.1.

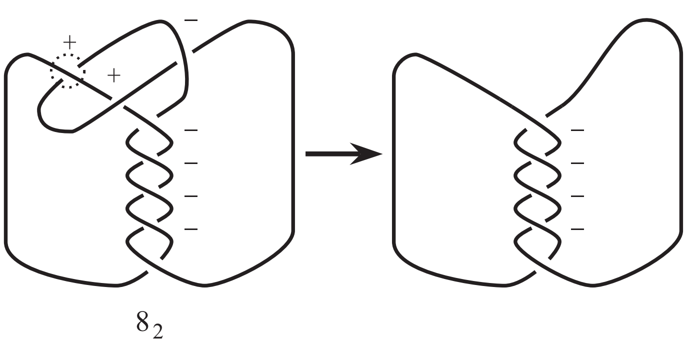
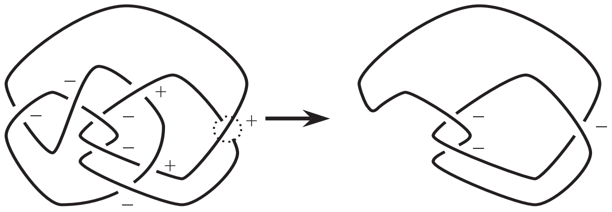


FIGURE 3

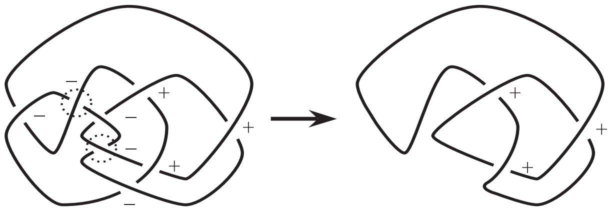
Example 3.3. Next, let K be the knot 8_{16} in Rolfsen's table (See Figure 4). Then $s(K) = \sigma(K) = -2$ and $u(K) = 2$. The knot K can be transformed into a negative knot as shown in Figure 3 by a single crossing change. So we know that $u_N(K) \leq 1$. On the other hand, we can show that $TB(K!) \neq s(K!) - 1$. Thus $u_N(K) = 1$ by Proposition 3.1.



8_{16}

Figure 4

The knot K can be also transformed into a positive knot as shown in Figure 5 by two crossing changes. So we know that $u_P(K) \leq 2$. Now let K^* be a knot obtained from K by a single crossing change. If $s(K^*) = 0$, then K^* is not a positive knot since K has unknotting number two and non-trivial positive knot has positive Rasmussen invariant by Theorem 2.2. If $s(K^*) \neq 0$, then $s(K^*)$ is still negative, because of Theorem 2.1(4). Hence K^* is not a positive knot by Theorem 1.1(1). Thus $u_P(K) \geq 2$ and we have $u_P(K) = 2$.



8_{16}

Figure 5

Finally, we show a list of the positive indices and the negative indices of knots with crossing number less than or equal 8 in Rolfsen's table. (The character X means that the number is one or two.)

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K	u_P	u_N	u	K	u_P	u_N	u
3_1	0	1	1	8_5	X	2	2
4_1	1	1	1	8_6	2	1	2
5_1	2	0	2	8_7	1	1	1
5_2	1	0	1	8_8	X	X	2
6_1	1	1	1	8_9	1	1	1
6_2	1	1	1	8_{10}	1	2	2
6_3	1	1	1	8_{11}	1	1	1
7_1	0	3	3	8_{12}	X	X	2
7_2	1	0	1	8_{13}	1	1	1
7_3	0	2	2	8_{14}	1	1	1
7_4	0	2	2	8_{15}	0	2	2
7_5	0	2	2	8_{16}	2	1	2
7_6	1	1	1	8_{17}	1	1	1
7_7	1	1	1	8_{18}	1	1	2
8_1	1	1	1	8_{19}	0	3	3
8_2	2	1	2	8_{20}	1	1	1
8_3	X	X	2	8_{21}	1	1	1
8_4	2	X	2				

Figure 6