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An infinite family of prime satellite knots with the same Alexander polynomial

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ABSTRACT

From Thurston's hyperbolization theorem [5], the set of classical knots are classified into torus knots, hyperbolic knots and satellite knots. In this paper, we show that for any prime knot K, there exists an infinite family $\{K_i\}$ of prime satellite knots such that K_i has the same Alexander polynomial as that of K, the knot K is the preimage of K_i and the knot group of K_i admits surjective homomorphism onto the knot group of K.

Keywords: satellite knot; knot group; Alexander polynomial

1. Introduction

Two knots K_1 and K_2 in 3-sphere S^3 are *equivalent* if there exists a homeomorphism $h : S^3 \to S^3$ such that $h(K_1) = K_2$. A *knot group* is the fundamental group of the complement of a knot K, denoted by $\pi_1(K)$. If two knots are equivalent then they have isomorphic groups. For given knots K and K', if there exists a surjective homomorphism from $\pi_1(K)$ to $\pi_1(K')$, then the *Alexander polynomial* [1][6] of K is divisible by that of K' (See [8].) A knot K is *composite* if it is a connected sum of two nontrivial knots. A knot is *prime* if it is nontrivial and not composite. P. R. Cromwell showed that there exist some families of prime satellite knots with the same Alexander polynomial [3]. Let $\Delta_t(K)$ be the Alexander polynomial of a knot K. In this paper, we consider the following problem.

Problem. For any knot K, are there infinitely many satellite knots $\{K_i\}$ such that $\Delta_t(K_i) = \Delta_t(K)$ and there exists surjective homomorphism $\pi_1(K_i) \to \pi_1(K)$?

We have the following result.

Theorem 1.1. *For any prime knot K^P, there exists an infinite family $\{K_i\}$ of (non-equivalent) prime satellite knots such that K^P is the preimage of K_i, $\Delta_t(K_i) = \Delta_t(K^P)$ and there exists a surjective homomorphism $\pi_1(K_i) \to \pi_1(K^P)$.*

Corollary 1.2. *There exists an infinite family $\{K_n\}$ of (non-equivalent) prime satellite knots with trivial Alexander polynomial.*

C. McA. Gordon and J. Luecke showed that the following theorem [4].

Theorem 1.3. *If two prime knots have isomorphic groups then they are equivalent.*

By making use of Theorems 1.1 and 1.3, we have the following.

Corollary 1.4. *For any prime knot K, there exists an infinite family of (non-isomorphic) knot groups of prime satellite knots with the same Alexander polynomial as $\Delta_t(K)$.*

2. Proofs of Theorem 1.1 and Corollary 1.2

Notation. For a manifold M, ∂M and $\overline{M}$ denote the boundary and the closure of M and N_M denotes the regular neighborhood of M.

Definititon 2.1. Let L be a link in a standardly embedded solid torus V in S^3 such that L is not contained a 3-ball in V and not isotopic to a core γ of V. Let $h_0 : V \to S^3$ (resp. $h_1 : V \to S^3$) be an embedding such that $h_0(\gamma)$ is knotted (resp. unknotted) in S^3. Let $T = h_0(\partial V)$. We say that $h_0(\gamma)$ is the *companion* of $h_0(L)$ for T, $h_0(L)$ is the *satellite* of $h_0(\gamma)$ for T and $h_1(L)$ is a *preimage* of $h_0(L)$ for T.

Proof of Theorem 1.1. Let K^P be an oriented prime knot in S^3. By making use of Lemma 1 [7], there exists an embedded disk D in S^3 which satisfies the following properties.
(i) $\partial D \cap K^P = \emptyset$,
(ii) K^P intersects D as in Figure 1,
(iii) $(N_D, N_D \cap K^P)$ is a trivial tangle,
(iv) $(\overline{S^3 - N_D}, \overline{(S^3 - N_D) \cap K^P})$ is a prime tangle.

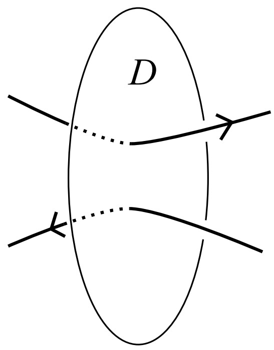

Figure 1

Put $(B_0, T^P) = (\overline{S^3 - N_D}, \overline{(S^3 - N_D) \cap K^P})$. By the property of the disk D, we know that there exists a standardly embedded solid torus V in S^3 such that D is a meridian disk of V and the interior of V contains K^P.

Now we need the following lemma.

Lemma 2.2. The prime knot K^P is not either contained in a 3-ball in V or isotopic to the core of V.

Proof. Because K^P is nontrivial, K^P cannot be isotopic to the core of V. We assume that the knot K^P is contained in a 3-ball in V. Then there exists a meridian disk D^* of V such that $D^* \cap K = \emptyset$. By using the ambient isotopy, we may assume that ∂D and ∂D^* are parallel on ∂V and $\partial D^* \cap N_D = \emptyset$. Now we can eliminate any intersection of D and D^* by using innermost argument and 2-handle surgeries of D^*. So we may assume that $N_D \cap D^* = \emptyset$. This is contradict to the assumption that (B_0, T^P) is a prime tangle since two arcs are separated by two disk D^*.

For a nontrivial knot K, let $\sharp_{i=0}^n K$ be a connected sum of nK's. (We assume that $\sharp_{i=0}^n K$ is a trivial knot if $n = 0$.) Let (B_1, T_n) be a doubled tangle of $\sharp_{i=0}^n K$ and let $\hat{V}$ be a regular neighbourhood of $\sharp_{i=0}^n K$. By Lemma 2.2, we can choose a homeomorphism $h : V \to \hat{V} \subset S^3$ such that h maps the (preferred) longitude of V to that of $\hat{V}$. We denote the resultant satellite knot $h(K^P)$ by K_n. (Note that K_0 is equivalent to the knot K^P. See Figure 2 in the case when K is a trefoil knot.)
By making use of Theorem 1.10 [9], we know that each K_n is a prime knot. By making use of Proposition 8.23 [1], we can show that $\Delta_t(K_n) = \Delta_t(K_P)$ for each n. Note that K_m is the preimage of K_n if $n < m$.

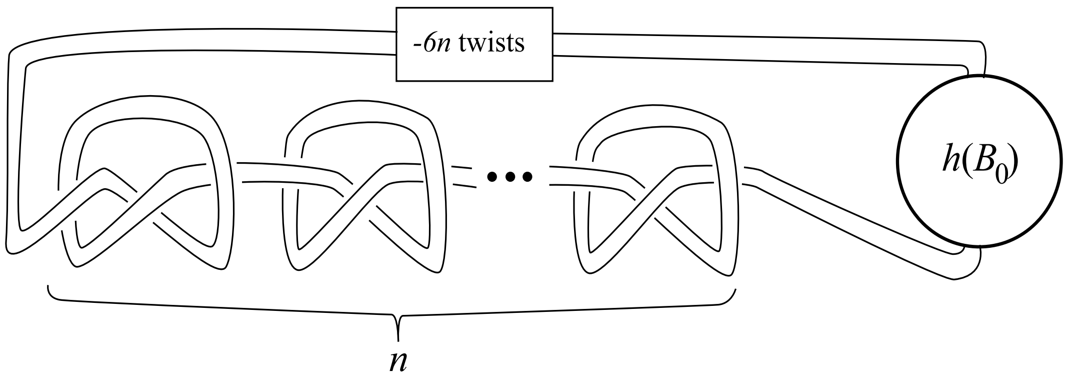

FIGURE 2

So by making use of Theorem 1 [10], if $n < m$, we know that K_m is not equivalent to K_n. By the construction and by making use of the idea in the proof of Theorem 1 [2], we can show that there exists a surjective homomorphism $\pi_1(K_n) \to \pi_1(K_0)$. This completes the proof.

Proof of Corollary 1.2. Let K be a prime knot with trivial Alexander polynomial (for example, a *pretzel knot* $\varphi(-3, 5, 7)$ as in Figure 3). By Theorem 1.1, there exists an infinite family $\{K_n\}$ of prime satellite knots such that K_n has trivial Alexander polynomial.

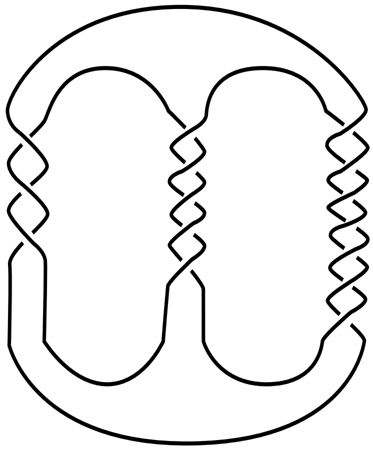

FIGURE 3

3. EXAMPLE

In this section, we give an example of a prime satellite knot with trivial Alexander polynomial.

Example 3.1. Let L be a knot which is called *Kinoshita-Terasaka knot* in a standardly embedded solid torus V in S^3 as in Figure 4. This is an example of a knot with trivial Alexander polynomial.
Let K be a nontrivial knot in S^3. Let $h_0 : V \to N_K$ be an embedding such that $h_0(V) = N_K$ and h_0 maps the preferred longitude of V to that of N_K. Let $\hat{K}$ be $h_0(L)$. Then we say that $\hat{K}$ is a *KT-double of K*. (Figure 5 describes a KT-double of a trefoil knot.) We know that a KT-double of K is prime. For example, the dotted circle shown in Figure 5 divides $\hat{K}$ into two prime tangles. By making use of the

same method in the proof of Theorem 1.1, we can show that KT-double of any nontrivial knot K has trivial Alexander polynomial since L has trivial Alexander polynomial.

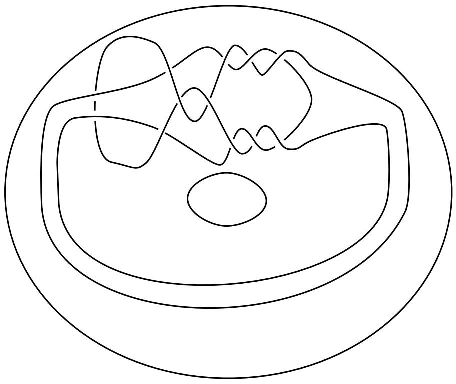

Figure 4

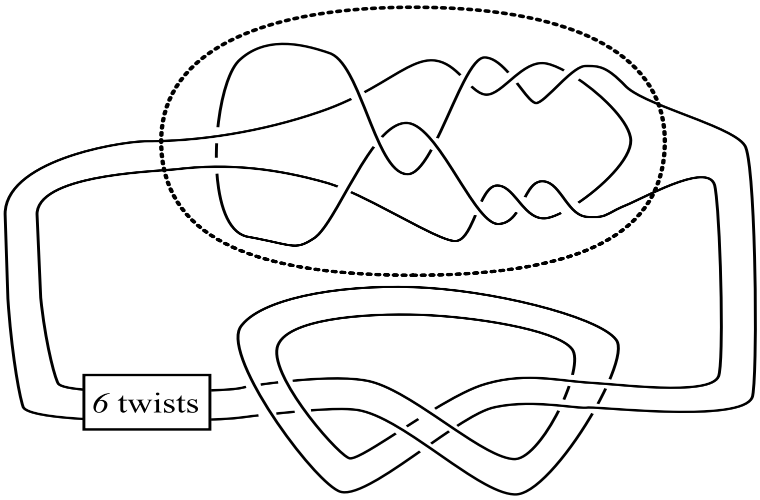

Figure 5

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