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The $L^{(\alpha)}$-harmonic extensions and their properties

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ABSTRACT. The $L^{(\alpha)}$-harmonic functions are solutions of the parabolic operator $L^{(\alpha)} = \partial/\partial t + (-\Delta_x)^\alpha$. In this paper, we study $L^{(\alpha)}$-harmonic extensions and their properties, which are defined by convolutions of the fundamental solution of $L^{(\alpha)}$. In particular, we show convergence theorems of the $L^{(\alpha)}$-harmonic extensions with respect to natural topology.

1. Introduction

Let $n \geq 1$ and let H be the upper half-space of the $(n+1)$-dimensional Euclidean space, that is, $H = \{X = (x,t) \in \mathbb{R}^{n+1} : x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha, \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $\Delta_x = \partial_1^2 + \cdots + \partial_n^2$. Let $C(H)$ be the set of all real-valued continuous functions on H. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions (for details, see Section 3). In this paper, we study $L^{(\alpha)}$-harmonic extensions on H and their properties, which are defined by convolutions of the fundamental solution of $L^{(\alpha)}$. And, we also show convergence theorems of the $L^{(\alpha)}$-harmonic extensions.

To state our main results, we give some definitions. We denote by $W^{(\alpha)}$ the fundamental solution of $L^{(\alpha)}$ (see Section 2 for the definition). For $1 \leq p < \infty$, the Lebesgue space $L^p := L^p(\mathbb{R}^n)$ is defined to be the Banach space of Lebesgue measurable (real-valued) functions f on $\mathbb{R}^n$ with

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dV_n(x)\right)^{1/p} < \infty,$$

where dV_n is the Lebesgue measure on $\mathbb{R}^n$. Furthermore, the Lebesgue space $L^\infty := L^\infty(\mathbb{R}^n)$ is defined to be the Banach space of Lebesgue measurable (real-valued) functions f on $\mathbb{R}^n$ with

$$\|f\|_{L^\infty} := \text{ess sup}\{|f(x)| \,:\, x \in \mathbb{R}^n\} < \infty.$$

For $1 \leq p \leq \infty$, we define an $L^{(\alpha)}$-harmonic extension $\mathcal{H}_f^{(\alpha)}$ of $f \in L^p$ by

$$\mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t) f(y) dV_n(y), \qquad (x,t) \in H. \tag{1.2}$$

Let $M := M(\mathbb{R}^n)$ be the set of all signed Borel measures on $\mathbb{R}^n$ (thus necessarily, $|\mu|(\mathbb{R}^n) < \infty$ for all $\mu \in M$). We also define an $L^{(\alpha)}$-harmonic extension $\mathcal{H}_\mu^{(\alpha)}$ of $\mu \in M$ by

$$\mathcal{H}_\mu^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t) d\mu(y), \qquad (x,t) \in H. \tag{1.3}$$

We shall show that the functions $\mathcal{H}_f^{(\alpha)}$ and $\mathcal{H}_\mu^{(\alpha)}$ are $L^{(\alpha)}$-harmonic on H (see Section 5). It is well known that when $\alpha = 1/2$, the fundamental solution $W^{(1/2)}$ coincides with the Poisson kernel for H (see [2, Section 2]). Therefore, the functions $\mathcal{H}_f^{(1/2)}$ and $\mathcal{H}_\mu^{(1/2)}$ are the usual harmonic extensions of f and μ, respectively. Let $C_c := C_c(\mathbb{R}^n)$ be the set of all continuous functions on $\mathbb{R}^n$ with compact support, and let $C_0 := C_0(\mathbb{R}^n)$ be the set of all continuous functions on $\mathbb{R}^n$ that vanish at ∞. Furthermore, we denote by $C_c^\infty := C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. We recall that the Banach spaces L^∞ and M are, respectively, the dual spaces of L^1 and C_0. We present main results of this paper. When $\alpha = 1/2$ (that is, in the

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harmonic function theory), the following results are well known (for example, see [1, Chapter 7]). In this paper, we generalize results of usual harmonic extensions to that of $L^{(\alpha)}$-harmonic extensions for all $0 < \alpha \leq 1$.

THEOREM 1.1. *Let $0 < \alpha \leq 1$. Then, the following statements hold:*

(1) *If f is bounded and uniformly continuous on $\mathbb{R}^n$, then the functions $\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)$ converge to f uniformly on $\mathbb{R}^n$ as $t \to +0$.*

(2) *If $1 \leq p < \infty$ and $f \in L^p$, then the functions $\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)$ converge to f in the norm topology on L^p as $t \to +0$.*

(3) *If $f \in L^\infty$, then the functions $\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)$ converge to f in the weak-star topology on L^∞ as $t \to +0$.*

(4) *If $\mu \in M$, then the measures $\mathcal{H}_\mu^{(\alpha)}(\,\cdot\,, t)dV_n$ converge to μ in the weak-star topology on M as $t \to +0$.*

Furthermore, we shall show the following theorem.

THEOREM 1.2. *Let $0 < \alpha \leq 1$. For $1 \leq p \leq \infty$ and $f \in L^p$, the function $\mathcal{H}_f^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H. Furthermore, for $\mu \in M$, the function $\mathcal{H}_\mu^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H.*

We describe the construction of this paper. In Section 2, we recall the definition of approximate-identity properties, and we exhibit general theory concerning functions defined a kernel that satisfies approximate-identity properties. In Section 3, we observe basic properties of the $L^{(\alpha)}$-harmonic functions. In particular, we show that the operator $(-\Delta_x)^\alpha$ is well defined for all continuous functions ψ on H with compact support, and we also give an estimate of $(-\Delta_x)^\alpha\psi$. In Section 4, we present the explicit definition of the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$. Also, we show that $W^{(\alpha)}$ satisfies approximate-identity properties. Consequently, we show Theorem 1.1 (1). In Section 5, we show Theorem 1.1 (2), (3) and (4). We also show Theorem 1.2.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. The approximate-identity properties

In this section, we recall the definition of approximate-identity properties, and we exhibit general theory concerning functions defined a kernel that satisfies approximate-identity properties. We say that a Borel measurable function K on H satisfies approximate-identity properties if the function K holds the following properties:

(P.1)
$$K(x,t) > 0, \qquad (x,t) \in H.$$

(P.2)
$$\int_{\mathbb{R}^n} K(x,t)dV_n(x) = 1, \qquad 0 < t < \infty.$$

(P.3) For every $a \in \mathbb{R}^n$ and every $\delta > 0$,

$$\int_{|y-a|\geq\delta} K(x-y,t)dV_n(y) \to 0$$

as $(x,t) \to (a,0)$.

Furthermore, for $1 \leq p \leq \infty$ and $f \in L^p$, we define a function $\mathcal{H}_f^K$ on H by

$$\mathcal{H}_f^K(x,t) = \int_{\mathbb{R}^n} K(x-y,t)f(y)dV_n(y), \qquad (x,t) \in H. \tag{2.1}$$

For $\mu \in M$, we also define a function $\mathcal{H}_\mu^K$ on H by

$$\mathcal{H}_\mu^K(x,t) = \int_{\mathbb{R}^n} K(x-y,t)d\mu(y), \qquad (x,t) \in H. \tag{2.2}$$

In this section, we study general theory concerning functions $\mathcal{H}_f^K$ and $\mathcal{H}_\mu^K$.

THEOREM 2.1. *Suppose that a Borel measurable function K on H satisfies approximate-identity properties. Then, the following statements hold:*

(1) *Let $1 \le p \le \infty$ and $f \in L^p$. Then,*

$$\|\mathcal{H}_f^K(\,\cdot\,,t)\|_{L^p} \le \|\mathcal{H}_{|f|}^K(\,\cdot\,,t)\|_{L^p} \le \|f\|_{L^p} \tag{2.3}$$

for all $t > 0$.

(2) *Let $\mu \in M$. Then,*

$$\|\mathcal{H}_\mu^K(\,\cdot\,,t)\|_{L^1} \le \|\mathcal{H}_{|\mu|}^K(\,\cdot\,,t)\|_{L^1} \le \|\mu\| \tag{2.4}$$

for all $t > 0$, where $\|\mu\|$ denotes the total variation of μ.

(3) *If f is bounded and uniformly continuous on $\mathbb{R}^n$, then the functions $\mathcal{H}_f^K(\,\cdot\,,t)$ converge to f uniformly on $\mathbb{R}^n$ as $t \to +0$.*

PROOF. (1) Since the first inequality of (2.3) is clear, we only show the second inequality of (2.3). Let $1 \le p < \infty$. Then, by (P.1) and (P.2), the Jensen inequality and the Fubini theorem imply that

$$\begin{aligned}
\|\mathcal{H}_{|f|}^K(\,\cdot\,,t)\|_{L^p}^p &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y,t)|f(y)|dV_n(y) \right)^p dV_n(x) \\
&\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y,t)|f(y)|^p dV_n(y) dV_n(x) \\
&= \int_{\mathbb{R}^n} |f(y)|^p \int_{\mathbb{R}^n} K(x-y,t) dV_n(x) dV_n(y) \\
&= \|f\|_{L^p}^p.
\end{aligned}$$

The case $p = \infty$ is easier than the case $1 \le p < \infty$. In fact, by (P.1) and (P.2), we have

$$|\mathcal{H}_{|f|}^K(x,t)| = \int_{\mathbb{R}^n} K(x-y,t)|f(y)|dV_n(y) \le \|f\|_{L^\infty}.$$

(2) Let $\mu \in M$. Since the first inequality of (2.4) is clear, we only show the second inequality of (2.4). By (P.1) and (P.2), the Fubini theorem implies that

$$\begin{aligned}
\|\mathcal{H}_{|\mu|}^K(\,\cdot\,,t)\|_{L^1} &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y,t) d|\mu|(y) dV_n(x) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y,t) dV_n(x) d|\mu|(y) \\
&= \|\mu\|.
\end{aligned}$$

(3) By (P.1) and (P.2), we have

$$\begin{aligned}
|\mathcal{H}_f^K(x,t) - f(x)| &= \left| \int_{\mathbb{R}^n} K(x-y,t)\big(f(y)-f(x)\big) dV_n(y) \right| \\
&\le \int_{\mathbb{R}^n} K(x-y,t)\big|f(y)-f(x)\big| dV_n(y).
\end{aligned}$$

for all $x \in \mathbb{R}^n$. Let $\varepsilon > 0$ be arbitrary. Then, by the uniform continuity of f on $\mathbb{R}^n$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x,\ y \in \mathbb{R}^n$ with $|x-y| < \delta$. Therefore, by (P.1), we obtain

$$\begin{aligned}
&|\mathcal{H}_f^K(x,t) - f(x)| \\
&\le \int_{|x-y|<\delta} K(x-y,t)\big|f(y)-f(x)\big| dV_n(y) + \int_{|x-y|\ge\delta} K(x-y,t)\big|f(y)-f(x)\big| dV_n(y) \\
&\le \varepsilon \int_{|x-y|<\delta} K(x-y,t) dV_n(y) + 2\|f\|_{L^\infty} \int_{|x-y|\ge\delta} K(x-y,t) dV_n(y)
\end{aligned}$$

$$\leq \varepsilon + 2\|f\|_{L^\infty} \int_{|x-y|\geq\delta} K(x-y,t)dV_n(y)$$
$$= \varepsilon + 2\|f\|_{L^\infty} \int_{|z|\geq\delta} K(-z,t)dV_n(z)$$

for all $x \in \mathbb{R}^n$. Hence, (P.3) implies that

$$\lim_{t\to+0} \left(\sup_{x\in\mathbb{R}^n} |\mathcal{H}_f^K(x,t) - f(x)| \right) \leq \varepsilon.$$

Since ε is arbitrary, the desired result is obtained. □

3. The $L^{(\alpha)}$-harmonic functions

In this section, we observe basic properties concerning the $L^{(\alpha)}$-harmonic functions. We begin with recalling definition of the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C^\infty(H)$ be the set of all infinitely differentiable functions on H, and let $C_c^\infty(H)$ be the set of all functions in $C^\infty(H)$ with compact support. Then, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(3.1) \qquad (-\Delta_x)^\alpha \psi(x,t) := -C_{n,\alpha} \lim_{\varepsilon\to+0} \int_{|y|>\varepsilon} \frac{\psi(x+y,t)-\psi(x,t)}{|y|^{n+2\alpha}} dV_n(y)$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma\big((n+2\alpha)/2\big)/\Gamma(-\alpha) > 0$ and Γ is the gamma function. In the following proposition, we show that $(-\Delta_x)^\alpha\psi$ is well defined for all $\psi \in C_c^\infty(H)$, and we also give estimates of $(-\Delta_x)^\alpha\psi(x,t)$.

PROPOSITION 3.1. *Let $0 < \alpha \leq 1$ and $\psi \in C_c^\infty(H)$. Then, $(-\Delta_x)^\alpha\psi(x,t)$ is well defined for all $(x,t) \in H$. Furthermore, there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that*

$$(3.2) \qquad \operatorname{supp}\big((-\Delta_x)^\alpha\psi\big) \subset S := \mathbb{R}^n \times [t_1, t_2]$$

and

$$(3.3) \qquad |(-\Delta_x)^\alpha\psi(x,t)| \leq C(1+|x|)^{-n-2\alpha}, \qquad (x,t) \in S.$$

The proof relies on the following lemma.

LEMMA 3.2. *Let $f \in C_c^\infty$ and let $x \in \mathbb{R}^n$ be fixed. Put*

$$h(y) := f(x+y) - f(x) - \sum_{j=1}^n y_j \partial_j f(x), \qquad y = (y_1, \ldots, y_n) \in \mathbb{R}^n,$$

then, there exists a constant $C > 0$ such that

$$|h(y)| \leq C|y|^2$$

for all $y \in \mathbb{R}^n$.

PROOF. Let $f \in C_c^\infty$ and $x \in \mathbb{R}^n$. Then, by the Taylor theorem, for each $y \in \mathbb{R}^n$, there exists $0 < \theta_y < 1$ such that

$$f(x+y) - f(x) = \sum_{j=1}^n y_j\partial_j f(x) + \frac{1}{2!}\left(\sum_{j=1}^n y_j\partial_j\right)^2 f(x+\theta_y y)$$
$$= \sum_{j=1}^n y_j\partial_j f(x) + \frac{1}{2}\sum_{j,k=1}^n y_j y_k \partial_j\partial_k f(x+\theta_y y).$$

Since $f \in C_c^\infty$, we have

$$|h(y)| \le \frac{1}{2} \sum_{j,k=1}^{n} |y_j||y_k||\partial_j\partial_k f(x+\theta_y y)| \le \frac{1}{2}|y|^2 \sum_{j,k=1}^{n} \|\partial_j\partial_k f\|_{L^\infty} = C|y|^2$$

for all $y \in \mathbb{R}^n$. □

PROOF OF PROPOSITION 3.1. Since $\psi \in C_c^\infty(H)$, there exist $R > 0$ and $0 < t_1 < t_2 < \infty$ such that

$$\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^n : |x| \le R\} \times [t_1, t_2]. \tag{3.4}$$

Because the case $\alpha = 1$ is trivial, we show the case $0 < \alpha < 1$. Thus, let $0 < \alpha < 1$ and $(x, t) \in H$. If $t \notin [t_1, t_2]$, then $(-\Delta_x)^\alpha \psi(x, t) = 0$. Hence, (3.2) is clearly obtained. Therefore, let $t \in [t_1, t_2]$ be fixed. Then, it suffices to show that $(-\Delta_x)^\alpha \psi(x, t)$ is well defined and there exists a constant $C > 0$ such that the estimate of (3.3) holds. Put $f(z) := \psi(z, t)$ for $z \in \mathbb{R}^n$.

Suppose $|x| \le 2R$. Then, since the function f belongs to C_c^∞, Lemma 3.2 implies that

$$\begin{aligned}
&\lim_{\varepsilon \to +0} \int_{|y|>\varepsilon} \frac{\psi(x+y,t) - \psi(x,t)}{|y|^{n+2\alpha}} dV_n(y) \\
&= \lim_{\varepsilon \to +0} \int_{3R \ge |y| > \varepsilon} \frac{f(x+y) - f(x)}{|y|^{n+2\alpha}} dV_n(y) \\
&= \lim_{\varepsilon \to +0} \int_{3R \ge |y| > \varepsilon} \frac{\sum_{j=1}^n y_j \partial_j f(x) + h(y)}{|y|^{n+2\alpha}} dV_n(y) \\
&= \sum_{j=1}^{n} \partial_j f(x) \lim_{\varepsilon \to +0} \int_{3R \ge |y| > \varepsilon} \frac{y_j}{|y|^{n+2\alpha}} dV_n(y) + \lim_{\varepsilon \to +0} \int_{3R \ge |y| > \varepsilon} \frac{h(y)}{|y|^{n+2\alpha}} dV_n(y).
\end{aligned} \tag{3.5}$$

We remark that the first term of the right-hand side of (3.5) is equal to 0. We give an estimate of the second term of the right-hand side of (3.5). Here, we note the polar coordinates formula for integration on $\mathbb{R}^n$. Let S_n be the unit sphere of $\mathbb{R}^n$, that is, the boundary of the unit ball B_n of $\mathbb{R}^n$. Furthermore, we denote by $d\sigma$ the normalized surface-area measure on S_n. Then, for a Borel measurable, integrable function g on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} g(x) dV_n(x) = n V_n(B_n) \int_0^\infty r^{n-1} \int_{S_n} g(r\zeta) d\sigma(\zeta)\, dr. \tag{3.6}$$

Thus, we have

$$\begin{aligned}
&\int_{3R \ge |y| > \varepsilon} \frac{|h(y)|}{|y|^{n+2\alpha}} dV_n(y) \le C \int_{3R \ge |y| > \varepsilon} |y|^{2-n-2\alpha} dV_n(y) \\
&= C \int_\varepsilon^{3R} r^{n-1} \int_{S_n} |r\zeta|^{2-n-2\alpha} d\sigma(\zeta)\, dr = C \int_\varepsilon^{3R} r^{1-2\alpha}\, dr \\
&= C \frac{(3R)^{2(1-\alpha)} - \varepsilon^{2(1-\alpha)}}{2(1-\alpha)}
\end{aligned}$$

for all $\varepsilon > 0$. Since $0 < \alpha < 1$, we obtain the convergence of the second term of the right-hand side of (3.5). Hence, $(-\Delta_x)^\alpha \psi(x, t)$ is well defined and (3.3) holds for the suitable constant C.

Suppose $|x| > 2R$. Then, by (3.4), we have $\operatorname{supp}(f) \subset \{x \in \mathbb{R}^n : |x| \le R\}$. Hence, we obtain

$$\begin{aligned}
&\lim_{\varepsilon \to +0} \int_{|y|>\varepsilon} \frac{\psi(x+y,t) - \psi(x,t)}{|y|^{n+2\alpha}} dV_n(y) = \lim_{\varepsilon \to +0} \int_{|y|>\varepsilon} \frac{f(x+y)}{|y|^{n+2\alpha}} dV_n(y) \\
&= \lim_{\varepsilon \to +0} \int_{E(\varepsilon,R)} \frac{f(x+y)}{|y|^{n+2\alpha}} dV_n(y),
\end{aligned}$$

where

$$E(\varepsilon, R) = \{y \in \mathbb{R}^n : |y| > \varepsilon\} \cap \{y \in \mathbb{R}^n : |x+y| \le R\}.$$

If $y \in E(\varepsilon, R)$, then

$$|y| \ge |x| - |x+y| \ge |x| - R \ge \frac{1}{2}|x| = \frac{1}{4}|x| + \frac{1}{4}|x| \ge \frac{1}{2}R + \frac{1}{4}|x| \ge C(1+|x|),$$

where the constant C depends only on R. Therefore, we conclude

$$\int_{|y|>\varepsilon} \frac{|\psi(x+y,t)-\psi(x,t)|}{|y|^{n+2\alpha}} dV_n(y) = \int_{|y|>\varepsilon} \frac{|f(x+y)|}{|y|^{n+2\alpha}} dV_n(y)$$
$$\leq C(1+|x|)^{-n-2\alpha} \int_{E(\varepsilon,R)} |f(x+y)| dV_n(y) \leq C(1+|x|)^{-n-2\alpha} \|f\|_{L^\infty} \int_{|z|\leq R} dV_n(z)$$

for all $\varepsilon > 0$ and $x \in \mathbb{R}^n$ with $|x| > 2R$. Hence, $(-\Delta_x)^\alpha \psi(x,t)$ is well defined and (3.3) holds for all $x \in \mathbb{R}^n$ with $|x| > 2R$. This completes the proof of proposition. □

We give definition of the $L^{(\alpha)}$-harmonic functions.

DEFINITION 3.3. Let $\widetilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is,

$$\int_H |u \cdot \widetilde{L}^{(\alpha)}\psi| dV_{n+1} < \infty \quad \text{and} \quad \int_H u \cdot \widetilde{L}^{(\alpha)}\psi dV_{n+1} = 0$$

for all $\psi \in C_c^\infty(H)$.

By Proposition 3.1, we give the following theorem.

THEOREM 3.4. *Let* $0 < \alpha \leq 1$. *Then, the following statements hold:*

(1) *Let* $\psi \in C_c^\infty(H)$. *Then,* $\widetilde{L}^{(\alpha)}\psi(x,t)$ *is well defined for all* $(x,t) \in H$. *Furthermore, there exist* $0 < t_1 < t_2 < \infty$ *and a constant* $C > 0$ *such that*

$$\operatorname{supp}(\widetilde{L}^{(\alpha)}\psi) \subset S = \mathbb{R}^n \times [t_1, t_2] \tag{3.7}$$

and

$$|\widetilde{L}^{(\alpha)}\psi(x,t)| \leq C(1+|x|)^{-n-2\alpha}, \qquad (x,t) \in S. \tag{3.8}$$

(2) *If a function* $u \in C(H)$ *satisfies*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dV_n(x) dt < \infty \tag{3.9}$$

for all $0 < t_1 < t_2 < \infty$, *then*

$$\int_H |u \cdot \widetilde{L}^{(\alpha)}\psi| dV_{n+1} < \infty$$

for all $\psi \in C_c^\infty(H)$.

4. The fundamental solution of $L^{(\alpha)}$

In this section, we present the explicit definition of the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ (for details, see [2]). Furthermore, we describe properties of $W^{(\alpha)}$, and also show that $W^{(\alpha)}$ satisfies approximate-identity properties. Consequently, we show Theorem 1.1 (1).

For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x \cdot \xi)\, dV_n(\xi) & (t > 0) \\ 0 & (t \leq 0), \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is called the fundamental solution of $L^{(\alpha)}$ and it is $L^{(\alpha)}$-harmonic on H.

In case $\alpha = 1/2$, the function $W^{(1/2)}$ is the Poisson kernel, that is,

$$W^{(1/2)}(x,t) = \begin{cases} \dfrac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \dfrac{t}{(|x|^2+t^2)^{(n+1)/2}} & (t>0) \\ 0 & (t \le 0). \end{cases}$$

In case $\alpha = 1$, the function $W^{(1)}$ is the Gauss kernel, that is,

$$W^{(1)}(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\Big(-\dfrac{|x|^2}{4t}\Big) & (t>0) \\ 0 & (t \le 0). \end{cases}$$

In other cases, simple explicit expressions for $W^{(\alpha)}$ are not known.

We describe basic properties of $W^{(\alpha)}$. It is well known that

$$W^{(\alpha)}(x,t) > 0, \qquad (x,t) \in H \tag{4.1}$$

and

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x,t)dV_n(x) = 1, \qquad 0 < t < \infty. \tag{4.2}$$

We also remark $W^{(\alpha)} \in C^\infty(H)$. The following estimate is [2, Lemma 3.1], which plays an important role in later argument: there exists a constant $C > 0$ such that

$$W^{(\alpha)}(x,t) \le C \frac{t}{(t+|x|^{2\alpha})^{n/(2\alpha)+1}} \tag{4.3}$$

for all $(x,t) \in H$.

We show the following proposition. Thus, $W^{(\alpha)}$ satisfies approximate-identity properties, that is, $W^{(\alpha)}$ satisfies (P.1), (P.2) and (P.3).

PROPOSITION 4.1. *Let $0 < \alpha \le 1$. Then, for every $a \in \mathbb{R}^n$ and every $\delta > 0$,*

$$\int_{|y-a|\ge\delta} W^{(\alpha)}(x-y,t)dV_n(y) \to 0 \tag{4.4}$$

as $(x,t) \to (a,0)$.

PROOF. By the translation on $\mathbb{R}^n$, we only prove the case $a = 0 \in \mathbb{R}^n$. Let $\delta > 0$. Then, by (4.3), we obtain

$$\begin{aligned} \int_{|y|\ge\delta} W^{(\alpha)}(x-y,t)dV_n(y) &\le C\int_{|y|\ge\delta} \frac{t}{(|x-y|^{2\alpha}+t)^{n/(2\alpha)+1}} dV_n(y) \\ &\le Ct\int_{|y|\ge\delta} \frac{1}{|x-y|^{n+2\alpha}} dV_n(y). \end{aligned}$$

Since it suffices to consider for points $x \in \mathbb{R}^n$ near $0 \in \mathbb{R}^n$, let $|x| \le \delta/2$. Then, for $y \in \mathbb{R}^n$ with $|y| \ge \delta$, we have

$$|x-y| \ge |y| - |x| \ge |y| - \frac{\delta}{2} \ge |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

Therefore, using polar coordinates, we obtain

$$\begin{aligned} &\int_{|y|\ge\delta} W^{(\alpha)}(x-y,t)dV_n(y) \le Ct\int_{|y|\ge\delta} \frac{1}{|y|^{n+2\alpha}} dV_n(y) \\ &\le Ct\int_\delta^\infty r^{n-1}\int_{S_n} \frac{1}{|r\zeta|^{n+2\alpha}} d\sigma(\zeta)\, dr = Ct\int_\delta^\infty r^{-1-2\alpha}dr = Ct \end{aligned}$$

for all $x \in \mathbb{R}^n$ with $|x| \le \delta/2$. Thus, (4.4) tends to 0 as $(x,t) \to (0,0)$. □

By Theorem 2.1 and Proposition 4.1, we have the following theorem. In particular, Theorem 4.2 (3) below gives Theorem 1.1 (1).

THEOREM 4.2. *Let $0 < \alpha \le 1$. Then, the following statements hold:*

(1) *Let $1 \le p \le \infty$ and $f \in L^p$. Then,*

$$\|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)\|_{L^p} \le \|\mathcal{H}_{|f|}^{(\alpha)}(\,\cdot\,,t)\|_{L^p} \le \|f\|_{L^p}$$

for all $t > 0$.

(2) *Let* $\mu \in M$. *Then,*

$$\|\mathcal{H}_{\mu}^{(\alpha)}(\,\cdot\,,t)\|_{L^1} \leq \|\mathcal{H}_{|\mu|}^{(\alpha)}(\,\cdot\,,t)\|_{L^1} \leq \|\mu\|$$

for all $t > 0$.

(3) *If* f *is bounded and uniformly continuous on* $\mathbb{R}^n$, *then the functions* $\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)$ *converge to* f *uniformly on* $\mathbb{R}^n$ *as* $t \to +0$.

5. The $L^{(\alpha)}$-harmonic extensions

In this section, we show Theorem 1.1 (2), (3) and (4). Furthermore, we show Theorem 2.1, that is, the functions $\mathcal{H}_f^{(\alpha)}$ and $\mathcal{H}_\mu^{(\alpha)}$ are $L^{(\alpha)}$-harmonic on H. First, we show Theorem 1.1 (2), (3) and (4).

THEOREM 5.1. *Let* $0 < \alpha \leq 1$. *Then, the following statements hold:*

(1) *If* $1 \leq p < \infty$ *and* $f \in L^p$, *then the functions* $\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)$ *converge to* f *in the norm topology on* L^p *as* $t \to +0$.

(2) *If* $f \in L^\infty$, *then the functions* $\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)$ *converge to* f *in the weak-star topology on* L^∞ *as* $t \to +0$.

(3) *If* $\mu \in M$, *then the measures* $\mathcal{H}_\mu^{(\alpha)}(\,\cdot\,,t)dV_n$ *converge to* μ *in the weak-star topology on* M *as* $t \to +0$.

PROOF. (1) First, we show Theorem 5.1 (1) when $f \in C_c$. Let $f \in C_c$. Then, there exists a constant $R > 0$ such that $\mathrm{supp}(f) \subset \{x \in \mathbb{R}^n : |x| \leq R\}$. Since f is bounded and uniformly continuous on $\mathbb{R}^n$, Theorem 4.2 (3) implies that the functions $\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)$ converge to f uniformly on $\mathbb{R}^n$ as $t \to +0$. Therefore, we have

$$\begin{aligned}
&\|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t) - f\|_{L^p}^p \\
&= \int_{|x|\leq 2R} |\mathcal{H}_f^{(\alpha)}(x,t) - f(x)|^p dV_n(x) + \int_{|x|>2R} |\mathcal{H}_f^{(\alpha)}(x,t) - f(x)|^p dV_n(x) \\
&\leq \sup_{x\in\mathbb{R}^n} |\mathcal{H}_f^{(\alpha)}(x,t) - f(x)|^p \int_{|x|\leq 2R} dV_n(x) + \int_{|x|>2R} |\mathcal{H}_f^{(\alpha)}(x,t)|^p dV_n(x),
\end{aligned} \tag{5.1}$$

and the first term of the right-hand side of (5.1) tends to 0 as $t \to +0$. Thus, it suffices to show that

$$\int_{|x|>2R} |\mathcal{H}_f^{(\alpha)}(x,t)|^p dV_n(x) \to 0$$

as $t \to +0$. Indeed, for $x \in \mathbb{R}^n$ with $|x| > 2R$, by (4.1) and (4.2), the Jensen inequality and (4.3) imply that

$$\begin{aligned}
&|\mathcal{H}_f^{(\alpha)}(x,t)|^p \leq \int_{|y|<R} W^{(\alpha)}(x-y,t)|f(y)|^p dV_n(y) \\
&\leq C\int_{|y|<R} \frac{t}{(|x-y|^{2\alpha}+t)^{n/(2\alpha)+1}} |f(y)|^p dV_n(y) \leq Ct\int_{|y|<R} \frac{1}{|x-y|^{n+2\alpha}} |f(y)|^p dV_n(y).
\end{aligned}$$

Furthermore, for $y \in \mathbb{R}^n$ with $|y| < R$, we have

$$|x-y| \geq |x| - R \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|.$$

Hence, we obtain

$$|\mathcal{H}_f^{(\alpha)}(x,t)|^p \leq Ct\int_{|y|<R} \frac{1}{|x|^{n+2\alpha}} |f(y)|^p dV_n(y) \leq C\|f\|_{L^\infty}^p t\frac{1}{|x|^{n+2\alpha}} = Ct\frac{1}{|x|^{n+2\alpha}}.$$

Therefore, we conclude

$$\int_{|x|>2R} |\mathcal{H}_f^{(\alpha)}(x,t)|^p dV_n(x) \leq Ct\int_{|x|>2R} \frac{1}{|x|^{n+2\alpha}} dV_n(x)$$

$$= Ct\int_{2R}^{\infty} r^{n-1}\int_{S_n}\frac{1}{|r\zeta|^{n+2\alpha}}d\sigma(\zeta)\,dr = Ct\int_{2R}^{\infty} r^{-1-2\alpha}dr = Ct \to 0$$

as $t \to +0$.

Next, we show Theorem 5.1 (1) when $f \in L^p$. Let $f \in L^p$ and $\varepsilon > 0$ be arbitrary. Then, since C_c is dense in L^p, there exists $g \in C_c$ such that $\|f - g\|_{L^p} < \varepsilon$. Hence, Theorem 4.2 (1) implies that

$$\begin{aligned}\|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t) - f\|_{L^p} &\le \|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t) - \mathcal{H}_g^{(\alpha)}(\,\cdot\,,t)\|_{L^p} \\ &\quad + \|\mathcal{H}_g^{(\alpha)}(\,\cdot\,,t) - g\|_{L^p} + \|g - f\|_{L^p} \\ &\le \|\mathcal{H}_g^{(\alpha)}(\,\cdot\,,t) - g\|_{L^p} + 2\varepsilon.\end{aligned}$$

Because $g \in C_c$, we have

$$\varlimsup_{t\to+0}\|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t) - f\|_{L^p} \le 2\varepsilon.$$

Since ε is arbitrary, the desired result is obtained.

(2) Let $f \in L^\infty$ and $g \in L^1$. Then, the Fubini theorem implies that

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\mathcal{H}_f^{(\alpha)}(x,t)dV_n(x) &= \int_{\mathbb{R}^n} g(x)\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)f(y)dV_n(y)dV_n(x) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)g(x)dV_n(x)\, f(y)dV_n(y).\end{aligned}$$

Since $W^{(\alpha)}(-z,t) = W^{(\alpha)}(z,t)$ for all $(z,t) \in H$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\mathcal{H}_f^{(\alpha)}(x,t)dV_n(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W^{(\alpha)}(y-x,t)g(x)dV_n(x)\, f(y)dV_n(y) \\ &= \int_{\mathbb{R}^n} \mathcal{H}_g^{(\alpha)}(y,t)f(y)dV_n(y).\end{aligned}$$

Hence, by Theorem 5.1 (1), we conclude

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} g(x)\mathcal{H}_f^{(\alpha)}(x,t)dV_n(x) - \int_{\mathbb{R}^n} g(x)f(x)dV_n(x)\right| \\ &\le \int_{\mathbb{R}^n} |\mathcal{H}_g^{(\alpha)}(x,t) - g(x)||f(x)|dV_n(x) \\ &\le \|\mathcal{H}_g^{(\alpha)}(\,\cdot\,,t) - g\|_{L^1}\cdot\|f\|_{L^\infty} \\ &\to 0\end{aligned}$$

as $t \to +0$.

(3) Let $\mu \in M$ and $g \in C_0$. Then, as in the proof of Theorem 5.1 (2), we have

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\mathcal{H}_\mu^{(\alpha)}(x,t)dV_n(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W^{(\alpha)}(y-x,t)g(x)dV_n(x)\, d\mu(y) \\ &= \int_{\mathbb{R}^n} \mathcal{H}_g^{(\alpha)}(y,t)d\mu(y).\end{aligned}$$

Since g is bounded and uniformly continuous on $\mathbb{R}^n$, Theorem 4.2 (3) implies that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} g(x)\mathcal{H}_\mu^{(\alpha)}(x,t)dV_n(x) - \int_{\mathbb{R}^n} g(x)d\mu(x)\right| \\ &\le \int_{\mathbb{R}^n} |\mathcal{H}_g^{(\alpha)}(x,t) - g(x)|d|\mu|(x) \\ &\le \sup_{x\in\mathbb{R}^n}|\mathcal{H}_g^{(\alpha)}(x,t) - g(x)|\cdot\|\mu\| \\ &\to 0\end{aligned}$$

as $t \to +0$. Thus, this completes the proof of theorem. □

We also show Theorem 1.2.

THEOREM 5.2. *Let $0 < \alpha \leq 1$. For $1 \leq p \leq \infty$ and $f \in L^p$, the function $\mathcal{H}_f^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H. Furthermore, for $\mu \in M$, the function $\mathcal{H}_\mu^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H.*

PROOF. Let $1 \leq p \leq \infty$ and $f \in L^p$. And, put $\varphi(x) = (1+|x|)^{-n-2\alpha}$ for $x \in \mathbb{R}^n$. First, we show

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)|f(y)|\varphi(x)dV_n(y)dV_n(x)dt < \infty \tag{5.2}$$

for all $0 < t_1 < t_2 < \infty$. Indeed, let q be the exponent conjugate to p. Then, by the Hölder inequality and Theorem 4.2 (1), we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)|f(y)|\varphi(x)dV_n(y)dV_n(x)dt \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \mathcal{H}_{|f|}^{(\alpha)}(x,t)\varphi(x)dV_n(x)dt \leq \int_{t_1}^{t_2} \|\mathcal{H}_{|f|}^{(\alpha)}(\,\cdot\,,t)\|_{L^p}\cdot\|\varphi\|_{L^q}\,dt \\
&\leq \int_{t_1}^{t_2} \|f\|_{L^p}\cdot\|\varphi\|_{L^q}\,dt = (t_2-t_1)\|f\|_{L^p}\cdot\|\varphi\|_{L^q}.
\end{aligned}$$

Furthermore, by the polar coordinates formula, we have

$$\begin{aligned}
\|\varphi\|_{L^q}^q &= \int_{\mathbb{R}^n}(1+|x|)^{-q(n+2\alpha)}dV_n(x) = C\int_0^\infty r^{n-1}\int_{S_n}(1+|r\zeta|)^{-q(n+2\alpha)}d\sigma(\zeta)\,dr \\
&= C\int_0^\infty \frac{r^{n-1}}{(1+r)^{q(n+2\alpha)}}\,dr \leq C\int_0^1 \frac{r^{n-1}}{(1+r)^{q(n+2\alpha)}}\,dr + C\int_1^\infty r^{n-1-q(n+2\alpha)}\,dr < \infty.
\end{aligned}$$

Therefore, we obtain (5.2). Thus, Theorem 3.4 (2) shows

$$\begin{aligned}
&\int_H |\mathcal{H}_f^{(\alpha)}(x,t)\cdot\widetilde{L}^{(\alpha)}\psi(x,t)|dV_{n+1}(x,t) \\
&\leq \int_H\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)|f(y)|\cdot|\widetilde{L}^{(\alpha)}\psi(x,t)|dV_n(y)dV_{n+1}(x,t) < \infty
\end{aligned} \tag{5.3}$$

for all $\psi \in C_c^\infty(H)$. Hence, by (5.3), the Fubini theorem and the $L^{(\alpha)}$-harmonicity of $W^{(\alpha)}$ imply that

$$\begin{aligned}
&\int_H \mathcal{H}_f^{(\alpha)}(x,t)\cdot\widetilde{L}^{(\alpha)}\psi(x,t)dV_{n+1}(x,t) \\
&= \int_H\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)f(y)dV_n(y)\cdot\widetilde{L}^{(\alpha)}\psi(x,t)dV_{n+1}(x,t) \\
&= \int_{\mathbb{R}^n}\int_H W^{(\alpha)}(x-y,t)\cdot\widetilde{L}^{(\alpha)}\psi(x,t)dV_{n+1}(x,t)\,f(y)dV_n(y) \\
&= 0
\end{aligned}$$

for all $\psi \in C_c^\infty(H)$. Consequently, the function $\mathcal{H}_f^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H.

As in the proof of the case $p = 1$, we also obtain the function $\mathcal{H}_\mu^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H for all $\mu \in M$. Thus, this completes the proof of theorem. □

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