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# Lie algebras of polynomial vector fields on the complex line C

Dedicated to Prof.S.Kobayshi on the occasion of his 77th birthday

by
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**Abstract**

The purposes of the present note are to consider Lie algebras of polynomial vector fields on the complex line C and to investigate its Lie algebra structure; the results obtained here are included in the first author's master thesis study, supervised by the second author, and are detailed there [3].

## 1 Introduction

Let $M$=C be the complex line with the standard coordinate $z$. A vector field $X$ on C is a *polynomial vector field*, abbreviated PVF(of degree $n \in \mathbb{N} \cup \{0\}$), if it is expressed as $X = f(z)\dfrac{\partial}{\partial z}$, where $f(z) = \sum_{j=0}^{n} a_j z^j$, $(a_j \in \mathbb{C}, 1 \leq j \leq n,\ a_n \neq 0)$ is a polynomial of degree $n$. Then for two PVFs $X = f\dfrac{\partial}{\partial z}, Y = g\dfrac{\partial}{\partial z}$, we have the usual bracket product $[X, Y] := (fg' - gf')\dfrac{\partial}{\partial z}$, which is again a PVF. Hence the set $\mathfrak{p}$ of PVF's on C with this bracket product is obviously an infinite dimensional complex Lie algebra, which is referred to the *polynomial Lie algebra* or the LAP. Any Lie subalgebra of $\mathfrak{p}$ may be called a LAP. Then we consider a LAP $\mathfrak{g}$ generated by finitely many $k$ polynomial vector fields $\{X_i; 1 \leq i \leq k\}$ of degree less than $n + 1$:

$$X_i = (\sum_{j=1}^{n} a_{ij} z^j)\frac{\partial}{\partial z},\ a_{ij} \in \mathbb{C}.$$

We investigate conditions for $\mathfrak{g}$ to be finite-dimensional and further we examine which one is abelian, nilpotent, solvable, or semisimple. The procedure of investigation is ad hoc and not comprehensive. If a LAP $\mathfrak{g}$ is finite dimensional we can define its degree $d$, denoted $d = \deg\, \mathfrak{g}$, by the highest degree of PVF in $\mathfrak{g}$. Then we have two numerical invariants, deg $\mathfrak{g}$ and dim $\mathfrak{g}$, closely related but differ in general. We first consider LAPs of degree less than 3 and have only three cases to investigate: $\dim = 1, \dim = 2$, and $\dim = 3$. The first case; $\mathfrak{g}$ is abelian and trivial, the second and the third are worth to examine. For higher degree cases we examine whether there exist LAPs of dim $\geq 2$. For further references we use the terminology *low dimension* for dimensions less than three, otherwise *higher dimension*. The readers are referred to [10] for the basics of Lie algebras of holomorphic vector fields on complex manifolds. We should remark that even if $\mathfrak{g}$ is generated by finitely many PVFs of degree less than $n$, it doesn't imply deg $\mathfrak{g} \leq n$, as is easily observed; deg $\mathfrak{g} \geq n$ in general.

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## 2 Lie algebras of quadratic vector fields

In this subsection we consider polynomial vector fields of degree two, which we call *quadratic vector fields*, or *QVF*. A linear vector field mean degree one PVF. If a C vector space $\mathfrak{g}$ is generated by finitely many QVF's, it is obvious dim $\mathfrak{g} \leq 3$, since the quadratic polynomials are generated by $1, z, z^2$. As for the bracket product of the Lie algebra structure, we have

$$[X_i, X_j] = ((a_{i1}a_{j2} - a_{i2}a_{j1})z^2 + 2(a_{i0}a_{j2} - a_{i2}a_{j0})z + (a_{i0}a_{j1} - a_{i1}a_{j0}))\frac{\partial}{\partial z}.$$

Thus we have $[X_i, X_j] \in \mathfrak{g}$, for all $X_i, X_j \in \mathfrak{g}$, which implies every element $X$ of $\mathfrak{g}$ is a linear or quadratic vector field. So we can define a quadratic Lie algebra $\mathfrak{g}$ to be a LAP of degree two. Furthermore it is of dimension less than 4. We define the *generic LAQ* to be $\mathfrak{g}$ generated by the base $\{1, z, z^2\}\frac{\partial}{\partial z}$.

### 2.1 Three dimensional quadratic Lie algebra

In this subsection we discuss the generic LAQ, *i.e.*, three dimensional case. Before stating a result we recall the definition of the (first) derived ideal $D_\bullet := [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. Then we have the following

**Theorem 2.1** *There exists a unique three dimensional LAQ, the generic* $\mathfrak{g}$ *which is semisimple.*

*Proof.* We know that $\mathfrak{g}$ is semi-simple iff $\mathfrak{g}$ satisfies the condition $\mathfrak{g}=D_\bullet$, which is verified by the arguments in the previous sections (see also [3]). The uniqueness is obvious from the construction of $\mathfrak{g}$. Q.E.D.

### 2.2 Two dimensional quadratic Lie algebras

Here we discuss the two dimensional case.

**Theorem 2.2** *The set $S$ of two dimensional LAQs is in one to one correspondence with an algebraic surface of the quasi projective variety $\mathbb{C}^2 \times \mathbb{C}\mathbf{P}^1$ defined by the following homogeneous polynomial of degree 4:*

$$(a_{12}a_{20} - a_{10}a_{22})^2 = (a_{10}a_{21} - a_{11}a_{20})(a_{11}a_{22} - a_{12}a_{21}),$$

*where an element $\mathfrak{g} \in S$ is generated by $X_1 = (a_{12}z^2 + a_{11}z + a_{10})\frac{\partial}{\partial z}$ and $X_2 = (a_{22}z^2 + a_{21}z + a_{20})\frac{\partial}{\partial z}$.*

*Proof.* The LAQ $\mathfrak{g} = < X_1, X_2 >_{\mathbb{C}}$ is of dimension two, iff there exist $\alpha, \beta \in \mathbb{C}$ such that

$$[X_1, X_2] = \alpha X_1 + \beta X_2$$

$$[X_1, X_2] = ((a_{11}a_{22} - a_{12}a_{21})z^2 + 2(a_{10}a_{22} - a_{12}a_{20})z + (a_{10}a_{21} - a_{11}a_{20}))\frac{\partial}{\partial z}$$

$$\alpha X_1 + \beta X_2 = ((\alpha a_{12} + \beta a_{22})z^2 + (\alpha a_{11} + \beta a_{21})z + (\alpha a_{10} + \beta a_{20}))\frac{\partial}{\partial z}.$$

Hence we have the following linear equations with respect to the variables $(\alpha, \beta)$

$$\begin{pmatrix} a_{12} & a_{22} \\ a_{11} & a_{21} \\ a_{10} & a_{20} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} \\ 2(a_{10}a_{22} - a_{12}a_{20}) \\ a_{10}a_{21} - a_{11}a_{20} \end{pmatrix}$$

.

For the above to have solutions it suffices to hold

$$\det \begin{pmatrix} a_{12} & a_{22} & a_{11}a_{22} - a_{12}a_{21} \\ a_{11} & a_{21} & 2(a_{10}a_{22} - a_{12}a_{20}) \\ a_{10} & a_{20} & a_{10}a_{21} - a_{11}a_{20}. \end{pmatrix} = 0 \ .$$

Hence we have $(a_{12}a_{20} - a_{10}a_{22})^2 = (a_{10}a_{21} - a_{11}a_{20})(a_{11}a_{22} - a_{12}a_{21})$. Q.E.D.
On the structures of other LAQs we will remark in the final section.

# 3 Polynomial Lie algebras of higher degree ($d \geq 3$)

In the previous section we determined the possible LAQs. The finite dimensionality follows from the definition as was shown. In this section we will determine possible (finite dimensional) LAPs of higher degree (*i.e.*deg$\geq$ 3). Henceforth we use the terminology LAP for the *proper LAP*, *i.e.*non quadratic one, if there is no fear of confusion.

## 3.1 Two-dimensional LAP $\mathfrak{g}$

Since any one dimensional LAP is abelian we first examine the condition for $\mathfrak{g}$ to be of dimension two. Let us take

$$X = (\sum_{k=0}^{n} a_k z^k)(\frac{\partial}{\partial z}) \ , \ Y = (\sum_{k=0}^{n} b_k z^k)(\frac{\partial}{\partial z})$$

and

$$\begin{pmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \end{pmatrix} = s \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_2 \end{pmatrix}.$$

Then $\mathfrak{g}$ is 2-dimensional, iff there exist $\alpha, \beta \in \mathbb{C}$ such that $[X, Y] = \alpha X + \beta Y$. By comparing the coefficients of each terms, if we assume $a_i = b_i = 0$ for $i < 1, n < i$, we have

$$\begin{pmatrix} \alpha a_n + \beta b_n \\ \vdots \\ \alpha a_{n-j} + \beta b_{n-j} \\ \vdots \\ \alpha a_2 + \beta b_2 \\ \alpha a_1 + \beta b_1 \\ \alpha a_0 + \beta b_0 \end{pmatrix} = \begin{pmatrix} (\alpha + s\beta) a_n \\ \vdots \\ (\alpha + s\beta) a_{n-j} \\ \vdots \\ (\alpha + s\beta) a_2 \\ \alpha a_1 + \beta b_1 \\ \alpha a_0 + \beta b_0 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} (2k-1-n) a_{n-k+1} b_k \\ \vdots \\ \sum_{k=1}^{n+j} (2k-j-1-n) a_{n-k+1} b_{k-j} \\ \vdots \\ \sum_{k=1}^{2n-2} (2k-2n+1) a_{n-k+1} b_{k-n+2} \\ \sum_{k=1}^{2n-1} (2k-2n) a_{n-k+1} b_{k-n+1} \\ \sum_{k=1}^{2n} (2k-2n-1) a_{n-k+1} b_{k-n} \end{pmatrix} \tag{1}$$

Here from (1) we have

$$\begin{aligned} (\alpha + s\beta) a_n &= \sum_{k=1}^{n} (2k-1-n) a_{n-k+1} b_k \\ &= (1-n) a_n b_1 + (n-1) a_1 b_n \\ &= (1-n) a_n b_1 + s(n-1) a_1 a_n \\ &= (n-1) a_n (s a_1 - b_1) \ . \end{aligned}$$

Hence

$$(\alpha + s\beta) = (n-1)(s a_1 - b_1). \tag{2}$$

We have also

$$\begin{aligned} (\alpha + s\beta) a_{n-j} &= \sum_{k=1}^{n+j} (2k-1-n-j) a_{n-k+1} b_{k-j} \quad (1 \le j \le n-2) \\ &= (j-1-n) a_{n-j+1} b_0 + (j+1-n) a_{n-j} b_1 \\ &\qquad + (n+1-j) a_0 b_{n-j+1} + (n-1-j) a_1 b_{n-j} \\ &= (n+1-j)(s a_0 - b_0) a_{n-j+1} + (n-1-j)(s a_1 - b_1) a_{n-j} \\ (n-1)(s a_1 - b_1) a_{n-j} &= (n+1-j)(s a_0 - b_0) a_{n-j+1} + (n-1-j)(s a_1 - b_1) a_{n-j}. \end{aligned}$$

Hence

$$(n+1-j)(s a_0 - b_0) a_{n-j+1} = j(s a_1 - b_1) a_{n-j}. \tag{3}$$

Hence from (2),(3), we have

$$\frac{(s a_0 - b_0)}{(s a_1 - b_1)} = \frac{n a_{n-1}}{a_n} = \cdots = \frac{j a_{n-j}}{(n+1-j) a_{n-j+1}} = \cdots = \frac{(n-2) a_2}{3 a_3} = t. \tag{4}$$

Letting (1)

$$\begin{pmatrix} a_n & s a_n \\ a_{n-1} & s a_{n-1} \\ \vdots & \vdots \\ a_2 & s a_2 \\ a_1 & b_1 \\ a_0 & b_0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} \tag{5}$$

we have from (4),

$$\frac{c_n}{a_n} = \frac{c_{n-1}}{a_{n-1}} = \cdots = \frac{c_2}{a_2}.$$

Thus for (5) to have solutions, we must have

$$\det \begin{pmatrix} a_n & b_n & (n-1)(sa_1 - b_1)a_n \\ a_1 & b_1 & 2(a_0b_2 - a_2b_0) \\ a_0 & b_0 & a_0b_1 - a_1b_0 \end{pmatrix} = 0.$$

So we have

$$\begin{aligned} &\det \begin{pmatrix} 1 & s & (n-1)(sa_1 - b_1) \\ a_1 & b_1 & 2(a_0b_2 - a_2b_0) \\ a_0 & b_0 & a_0b_1 - a_1b_0 \end{pmatrix} \\ =&b_1(a_0b_1 - a_1b_0) + 2sa_0a_2(sa_0 - b_0) + a_1b_0(n-1)(sa_1 - b_1) \\ &\qquad - a_0b_1(n-1)(sa_1 - b_1) - 2a_2b_0(sa_0 - b_0) - sa_1(a_0b_1 - a_1b_0) \\ =&(b_1 - sa_1)(a_0b_1 - a_1b_0) - 2a_2(b_0 - sa_0)(sa_0 - b_0) + (n-1)(a_0b_1 - a_1b_0)(b_1 - sa_1) \\ =&n(b_1 - sa_1)(a_0b_1 - a_1b_0) + 2a_2(b_0 - sa_0)^2 \\ =&0. \end{aligned}$$

Hence

$$\begin{aligned} n(b_1 - sa_1)(a_0b_1 - a_1b_0) + 2a_2(b_0 - sa_0)^2 &= 0 \\ 2a_2(sa_0 - b_0)\frac{sa_0 - b_0}{sa_1 - b_1} &= n(a_0b_1 - a_1b_0) \\ 2a_2(sa_0 - b_0)t &= n(a_0b_1 - a_1b_0) \\ na_0b_1 + (2ta_2 - na_1)b_0 &= 2tsa_0a_2 \end{aligned} \tag{6}$$

Also we have from (4), and from $\dfrac{sa_0 - b_0}{sa_1 - b_1} = t$

$$b_0 = tb_1 - s(ta_1 - a_0) \tag{7}$$

Hence from (6),(7)

$$\begin{aligned} na_0b_1 + (2ta_2 - na_1)(tb_1 - s(ta_1 - a_0)) &= 2sta_0a_2 \\ (na_0 + 2t^2a_2 - tna_1)b_1 &= (na_0 + 2t^2a_2 - tna_1)sa_1. \end{aligned}$$

We have to examine case by case:

(**case I**) $na_0 + 2t^2a_2 - tna_1 \neq 0$

$$b_1 = sa_1 \quad \text{and} \quad b_0 = sa_0$$

Hence we have

$$Y = sX,$$

which implies $\mathfrak{g}$ is one -dimensional.

**(case II)** $na_0 + 2t^2a_2 - tna_1 = 0$

Then from (6,7) we have

$$\begin{pmatrix} na_0 & 2ta_2 - na_1 \\ t & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 2tsa_0a_2 \\ s(ta_1 - a_0) \end{pmatrix}.$$

For this to have solutions we deduce from $\det \begin{pmatrix} na_0 & 2ta_2 - na_1 \\ t & -1 \end{pmatrix} = 0$, the rank of

$$\begin{pmatrix} na_0 & 2ta_2 - na_1 & 2tsa_0a_2 \\ t & -1 & s(ta_1 - a_0) \end{pmatrix} \tag{8}$$

must be 1, where

$$\begin{aligned} &\text{rank} \begin{pmatrix} na_0 & 2ta_2 - na_1 & 2tsa_0a_2 \\ t & -1 & s(ta_1 - a_0) \end{pmatrix} \\ =&\text{rank} \begin{pmatrix} tna_0 & 2t^2a_2 - na_1t & 2t^2sa_0a_2 \\ t & -1 & s(ta_1 - a_0) \end{pmatrix} \\ =&\text{rank} \begin{pmatrix} tna_0 - tna_0 & 2t^2a_2 - na_1t + na_0 & 2t^2sa_0a_2 - sna_0(ta_1 - a_0) \\ t & -1 & s(ta_1 - a_0) \end{pmatrix} \\ =&\text{rank} \begin{pmatrix} t & -1 & s(ta_1 - a_0) \\ 0 & 0 & 2t^2sa_0a_2 - sna_0(ta_1 - a_0) \end{pmatrix}. \end{aligned}$$

Hence we have only to verify (if $sa_0 \neq 0$)

$$2t^2sa_0a_2 - stna_0a_1 + sna_0^2 = 0,$$

which holds whenever $na_0 + 2t^2a_2 - tna_1 = 0$.
Hence the rank of (8) is always one, if $na_0 + 2t^2a_2 - tna_1 = 0$.
Hence from

$$\begin{aligned} 2t^2a_2 - tna_1 + na_0 &= 0 \\ tna_1 - na_0 &= 2t^2a_2 \\ ta_1 - a_0 &= \frac{2t^2}{n}a_2, \end{aligned}$$

we have

$$\begin{aligned} b_0 &= tb_1 - s(ta_1 - a_0) \\ tb_1 - b_0 &= s(ta_1 - a_0) \\ tb_1 - b_0 &= \frac{2t^2s}{n}a_2 \end{aligned} \tag{9}$$

Here we can choose $a_0, a_1, b_0$, and $b_1$ so that $b_0 \neq sa_0$, or $b_1 \neq sa_1$.
Hence we have

**Theorem 3.1** *The LAP* $\mathfrak{g} =< X, Y >_{\mathbb{C}}$ *of degree* $n$ *is two dimensional iff the following conditions are satisfied:*

1. $^{\exists}s \in \mathbb{C}\ s.t.\ \begin{pmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \end{pmatrix} = s \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_2 \end{pmatrix}$
2. $^{\exists}t \in \mathbb{C}\ s.t.\ \dfrac{a_{n-1}}{na_n} = \cdots = \dfrac{ja_{n-j}}{(n+1-j)a_{n-j+1}} = \cdots = \dfrac{(n-2)a_2}{3a_3} = t$
3. $na_0 + 2t^2a_2 - tna_1 = 0$
4. $tb_1 - b_0 = \dfrac{2t^2s}{n}a_2 \quad (b_0 \neq sa_0 \quad or \quad b_1 \neq sa_1)$

As a corollary to the above we can generalize Theorem 2.2 in the following way:

**Corollary 3.1** *The set* $S$ *of two dimensional LAPs (of* $d$=deg$\geq 3$*) is in one to one correspondence with an algebraic surface of the quasi projective variety* $\mathbb{C}^{d-1} \times \mathbb{C}\mathbf{P}^1$

*Proof.* Let us take generators $X = (\sum_{k=0}^{n} a_k z^k)(\frac{\partial}{\partial z}), Y = (\sum_{k=0}^{n} b_k z^k)(\frac{\partial}{\partial z})$ of LAP $\mathfrak{g}$ of degree $n$. We can represent $X, Y$ by the complex vectors $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$. Then we can assume $X$ or $a$ is *monic*, *i.e.*, the top coefficient is normalized; $a_n = 1$. Then we can assume (from the linear independence) $Y$ is of deg$\leq n-1$. Then from the first condition we have $b_{n-1} = \cdots = b_2 = 0$. Also we have $(b_1, b_0) \in \mathbb{CP}^1$. The vector $(a, b)$ virtually represents an element of $\mathbb{C}^{d-1} \times \mathbb{CP}^1$

### 3.2 Three dimensional LAP $\mathfrak{g}$

In the preceding sections we have investigated two dimensional LAP. As for the other cases we have the following

**Theorem 3.2** *There are no LAPs of* dim $\geq 3$.

*Proof.* Here we prove only the 3-dimensional case. For the other cases the readers are referred to [3]. now assume the contrary; dim$\mathfrak{g}$= 3. Let

$$X = (\sum_{k=0}^{n} a_k z^k)(\frac{\partial}{\partial z})\ ,\ Y = (\sum_{k=0}^{n} b_k z^k)(\frac{\partial}{\partial z})\ ,\ Z = (\sum_{k=0}^{n} c_k z^k)(\frac{\partial}{\partial z})$$

be a base of the vector space $\mathfrak{g}$. Then from the assumption $[X, Y], [X, Z], [Y, Z] \in \mathfrak{g}$, we have by the same reasoning in the previous section

$$^{\exists}s, t \in \mathbb{C} \text{ s.t. } \begin{pmatrix} b_n \\ \vdots \\ b_2 \end{pmatrix} = s \begin{pmatrix} a_n \\ \vdots \\ a_2 \end{pmatrix}\ ,\ \begin{pmatrix} c_n \\ \vdots \\ c_2 \end{pmatrix} = t \begin{pmatrix} a_n \\ \vdots \\ a_2 \end{pmatrix}.$$

From $[X, Y] \in \mathfrak{g}$, we have also

$$^{\exists}\alpha, \beta, \gamma \in \mathbb{C} \text{ s.t. } [X, Y] = \alpha X + \beta Y + \gamma Z$$

and letting

$$[X,Y] = (\sum_{k=0}^{n} d_k z^k)\frac{\partial}{\partial z},$$

we have

$$\begin{pmatrix} a_n & b_n & c_n \\ \vdots & \vdots & \vdots \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a_n & sa_n & ta_n \\ \vdots & \vdots & \vdots \\ a_2 & sa_2 & ta_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} d_n \\ \vdots \\ d_2 \\ d_1 \\ d_0 \end{pmatrix} \tag{10}$$

Since there are $\alpha, \beta, \gamma$ satisfying (10), we have

$$\frac{d_n}{a_n} = \frac{d_{n-1}}{a_{n-1}} = \cdots = \frac{d_2}{a_2} = (\alpha + s\beta + t\gamma),$$

and

$$d_{n-j} = \sum_{k=1}^{n+j}(2k-j-1-n)a_{n-k+1}b_{k-j} \quad (0 \le j \le n-2).$$

Hence we have

$$\begin{aligned} \frac{d_n}{a_n} &= (\alpha + s\beta + t\gamma) \\ (\alpha + s\beta + t\gamma)a_n &= \sum_{k=1}^{n}(2k-1-n)a_{n-k+1}b_k \\ &= (n-1)(sa_1 - b_1). \end{aligned}$$

Further we have

$$\begin{aligned} \frac{d_{n-j}}{a_{n-j}} &= (\alpha + s\beta + t\gamma) \\ (\alpha + s\beta + t\gamma)a_{n-j} &= \sum_{k=1}^{n+j}(2k-1-n-j)a_{n-k+1}b_{k-j} \\ &= (n+1-j)(sa_0 - b_0)a_{n-j+1} + (n-1-j)(sa_1 - b_1)a_{n-j}. \end{aligned} \tag{11}$$

And so from

$$(n-1)(sa_1 - b_1)a_{n-j} = (n+1-j)(sa_0 - b_0)a_{n-j+1} + (n-1-j)(sa_1 - b_1)a_{n-j},$$

we have

$$(n+1-j)(sa_0 - b_0)a_{n-j+1} = j(sa_1 - b_1)a_{n-j}$$

.

Hence

$$\frac{sa_0 - b_0}{sa_1 - b_1} = \frac{a_{n-1}}{na_n} = \cdots = \frac{ja_{n-j}}{(n+1-j)a_{n-j+1}} = \cdots = \frac{(n-2)a_2}{3a_3} \tag{12}$$

Similarly from $[X, Z] \in \mathfrak{g}$, we have

$$\frac{ta_0 - c_0}{ta_1 - c_1} = \frac{a_{n-1}}{na_n} = \cdots = \frac{ja_{n-j}}{(n+1-j)a_{n-j+1}} = \cdots = \frac{(n-2)a_2}{3a_3} \tag{13}$$

and from $[Y, Z] \in \mathfrak{g}$ we have

$$\frac{tb_0 - sc_0}{tb_1 - sc_1} = \frac{b_{n-1}}{nb_n} = \cdots = \frac{jb_{n-j}}{(n+1-j)b_{n-j+1}} = \cdots = \frac{(n-2)b_2}{3b_3} \tag{14}$$

Hence from (12, 13, 14), we have

$$\frac{sa_0 - b_0}{sa_1 - b_1} = \frac{ta_0 - c_0}{ta_1 - c_1} = \frac{tb_0 - sc_0}{tb_1 - sc_1} = u \tag{15}$$

Further from (15) we have

$$\frac{sa_0 - b_0}{sa_1 - b_1} = u \qquad \text{and} \qquad ub_1 - b_0 = s(ua_1 - a_0) \tag{16}$$

$$\frac{ta_0 - c_0}{ta_1 - c_1} = u \qquad \text{and} \qquad uc_1 - c_0 = t(ua_1 - a_0) \tag{17}$$

Here

$$\begin{aligned}
\det\begin{pmatrix} a_n & \cdots & a_2 & a_1 & a_0 \\ b_n & \cdots & b_2 & b_1 & b_0 \\ c_n & \cdots & c_2 & c_1 & c_0 \end{pmatrix} &= \det\begin{pmatrix} 1 & \cdots & 1 & a_1 & a_0 \\ s & \cdots & s & b_1 & b_0 \\ t & \cdots & t & c_1 & c_0 \end{pmatrix} \\
&= \det\begin{pmatrix} 0 & \cdots & 0 & 1 & a_1 & a_0 \\ 0 & \cdots & 0 & s & b_1 & b_0 \\ 0 & \cdots & 0 & t & c_1 & c_0 \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & a_1 & a_0 \\ s & b_1 & b_0 \\ t & c_1 & c_0 \end{pmatrix} \\
&= b_1c_0 + ta_1b_0 + sa_0c_1 - ta_0b_1 - b_0c_1 - sa_1c_0 \\
&= (sa_0 - b_0)c_1 - (sa_1 - b_1)c_0 + t(a_1b_0 - a_0b_1)
\end{aligned}$$

From (15) we have $sa_0 - b_0 = u(sa_1 - b_1)$

$$\begin{aligned}
&= u(sa_1 - b_1)c_1 - (sa_1 - b_1)c_0 + t(a_1b_0 - a_0b_1) \\
&= (sa_1 - b_1)(uc_1 - c_0) + t(a_1b_0 - a_0b_1) \\
&= t(sa_1 - b_1)(ua_1 - a_0) + t(a_1b_0 - a_0b_1) \quad (\text{from}(17)) \\
&= stua_1^2 - sta_1a_0 - ta_1(ub_1 - b_0) \\
&= stua_1^2 - sta_1a_0 - sta_1(ua_1 - a_0) \quad \text{from}(16) \\
&= 0.
\end{aligned}$$

Thus it is proved the C vectors $a = (a_n \cdots a_0)$, $b = (b_n \cdots b_0)$, and $c = (c_n \cdots c_0)$ are C-linearly dependent, hence a contradiction. Q.E.D.

### 3.3 Structure of two dimensional LAP/Q

Here we remark the structure of two dimensional LAP/Q $\mathfrak{g}$: when $\mathfrak{g}$ is commutative, solvable, nilpotent? We have mentioned the 3-dimensional case just before: the semisimpleness of $\mathfrak{g}$. The solvability of two dimensional $\mathfrak{g}$ is well known for general abstract Lie algebras. W give only the following remark without proof (see [3] for the details).

**Remark 3.1** Every two dimensional LAP $\mathfrak{g}$ is solvable but never nilpotent irrespective of its degree.

## 4 Concluding remarks

The CR version of the corresponding theory might be formulated, but there seem few works to exist in this direction except [1, 5, 6, 10, 11]. We can also consider similar problems in the complex affine or projective spaces of higher dimension. One could obtain some interesting results in this direction, which might give non-trivial examples non-linear rational or geomorphic actions there.

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