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Theorems for the *n* dimensional simplex which are generalizations of Ceva's theorem on the triangle

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Abstract. We show other generalizations of Ceva's theorem on the triangle to the n dimensional simplex than the theorems given in [2]. They are Theorems 1.1, 1.2, 2.1 and 3.1 below.

Let n be any integer>2. Let $\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n$ denote an arbitrary n dimensional simplex in Euclidean space $\mathbf{R}^n$, whose vertices are $\{\mathrm{P}_j|\, 0 \le j \le n\}$. Let μ denote the standard Euclidean measure (=volume) on $\mathbf{R}^n$. Let m denote the standard Euclidean measure on $\mathbf{R}^{n-1}$. Let T be any point in the interior of $\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n$. For each integer j with $0 \le j \le n$, write $C_j = \{\mathrm{P}_x \,|\, 0 \le x \le n, x \ne j\}$. For each integer j with $0 \le j \le n$, let H_j denote the $n-1$ dimensional hyper-plane containing all the points in C_j. For each integer j with $0 \le j \le n$, let P'_j denote the point of the intersection of line $\mathrm{P}_j\mathrm{T}$ and hyper-plane H_j. For arbitrary finite points $\{\mathrm{V}_j \,|\, 1 \le j \le k\}$ in Euclidean space $\mathbf{R}^n$, let $<\{\mathrm{V}_j \,|\, 1 \le j \le k\}>$ denote the minimum closed convex subset, of $\mathbf{R}^n$, containing all the points $\{\mathrm{V}_j \,|\, 1 \le j \le k\}$.

Section 1. Case of any integer $n \ge 2$.

Recall that $\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n$ denote an arbitrary n dimensional simplex in Euclidean space $\mathbf{R}^n$, whose vertices are $\{\mathrm{P}_j|\, 0 \le j \le n\}$. Let j be any integer with

$0 \le j \le n.$ Recall $C_j = \{\mathrm{P}_x \mid 0 \le x \le n, x \ne j\}$. Let r be any positive integer ≥ 2. Let $\{k_u\}_{u=1}^{r}$ be any sequence of positive integers with $n+1 = \sum_{u=1}^{r} k_u$ and $k_u \ge k_{u+1} \ge 2$ for all the integers $1 \le u \le r-1$. For each integer j with $0 \le j \le k_1 - 2$, write $E_j = C_j \cap C_{j+1}$. Write $E_{k_1-1} = C_{k_1} \cap C_0$. For each integer j with $k_1 \le j \le k_1 + k_2 - 2$, write $E_j = C_j \cap C_{j+1}$. Write $E_{k_1+k_2-1} = C_{k_1+k_2-1} \cap C_{k_1}$. Let s be any integer with $1 \le s \le r-1$. For each integer j with $\sum_{u=1}^{s} k_u \le j \le \sum_{u=1}^{s+1} k_u - 2$, write $E_j = C_j \cap C_{j+1}$. Put $a_s = \sum_{u=1}^{s+1} k_u$. Write $E_{a_s-1} = C_{a_s-1} \cap C_{a_{s-1}}$. Here we put $a_0 = k_1$. So, we have $\{E_j \mid 0 \le j \le n\}$. For each $0 \le j \le n$ write $\Upsilon_j = \{S \mid S \subset C_j, |S| = n-1, S \ne E_j\}$. Then we have $|\Upsilon_j| = n-1$. We obtain

Theorem 1.1. $\displaystyle \prod_{j=0}^{n} \left(\frac{\prod_{\xi \in \Upsilon_j} m(<\{\mathrm{P}'_j\} \cup \xi>)}{m(<\{\mathrm{P}'_j\} \cup E_j>)^{n-1}} \right) = 1.$

Proof of Theorem 1.1. By the same way as in the proof of Theorem 2 in [2] we have

$$\frac{\prod_{\xi \in \Upsilon_j} m(<\{\mathrm{P}'_j\} \cup \xi>)}{m(<\{\mathrm{P}'_j\} \cup E_j>)^{n-1}} = \frac{\prod_{\xi \in \Upsilon_j} \mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup \xi>)}{\mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup E_j>)^{n-1}} .$$

We obtain $\displaystyle \prod_{j=0}^{n} \Big(\prod_{\xi \in \Upsilon_j} \mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup \xi>) \Big) = \prod_{j=0}^{n} \mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup E_j>)^{n-1}$.

Theorem 1.1 is proven.

Theorem 1 in [2] is the case of $r = 1$ of the above Theorem 1.1.

Corollary of Theorem 1.1. *Let* $\tau = \begin{pmatrix} 0 & 1 & 2 & \cdots & n \\ \tau(0) & \tau(1) & \tau(2) & \cdots & \tau(n) \end{pmatrix}$ *be any element of* S_{n+1}. *Rewrite* P_j *for* $\mathrm{P}_{\tau(j)}$ *with all* $0 \le j \le n$ *and* P'_j *for* $\mathrm{P}'_{\tau(j)}$ *with all* $0 \le j \le n$. *Then apply Theorem 1.1. Then we have the equation for simplex* $\mathrm{P}_{\tau(0)}\mathrm{P}_{\tau(1)}\mathrm{P}_{\tau(2)} \cdots \mathrm{P}_{\tau(n)}$.

(We have simplex $P_0P_1P_2\cdots P_n = P_{\tau(0)}P_{\tau(1)}P_{\tau(2)}\cdots P_{\tau(n)}$. We can consider Theorem 1.1 for any order $(P_{\tau(0)}, P_{\tau(1)}, P_{\tau(2)}, \cdots, P_{\tau(n)}, P_{\tau(0)})$.)

We obtain also

Theorem 1.2. *Let* $\sigma = \begin{pmatrix} 0 & 1 & 2 & \cdots & n \\ \sigma(0) & \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$ *be any element of* S_{n+1} *that satisfies* $\sigma(j) \neq j$ *for all the integers* $0 \le j \le n$. *For any* $0 \le j \le n$, *put* $F_{\sigma,j} = \{P_x \mid 0 \le x \le n, x \neq j, x \neq \sigma(j)\}$, $\Delta_{\sigma,j} = <\{P'_j\} \cup F_{\sigma,j}>$, *and*

$\Xi_{\sigma,j} = \{S \mid S \subset C_j, |S| = n-1, S \neq F_{\sigma,j}\}$. *Then we have*

$$\prod_{j=0}^{n}\left(\frac{\prod_{\xi\in\Xi_{\sigma,j}} m(<\{P'_j\}\cup\xi>)}{m(\Delta_{\sigma,j})^{n-1}}\right) = 1.$$

Proof of Theorem 1.2. By the same way as in the proof of Theorem 2 in [2] we have

$$\frac{\prod_{\xi\in\Xi_{\sigma,j}} m(<\{P'_j\}\cup\xi>)}{m(\Delta_{\sigma,j})^{n-1}} = \frac{\prod_{\xi\in\Xi_{\sigma,j}} \mu(<\{T, P_j\}\cup\xi>)}{\mu(<\{T, P_j\}\cup\{P_x \mid 0 \le j \le n, x \neq j, x \neq \sigma(j)\}>)^{n-1}}.$$

We obtain

$$\prod_{j=0}^{n}\Big(\prod_{\xi\in\Xi_{\sigma,j}} \mu(<\{T, P_j\}\cup\xi>)\Big) = \prod_{j=0}^{n} \mu(<\{T, P_j\}\cup\{P_x \mid 0 \le j \le n, x \neq j, x \neq \sigma(j)\}>)^{n-1}.$$

Theorem 1.2 is proven. (Remark. We have easily $|\Xi_{\sigma,j}| = n-1$ for any $0 \le j \le n$.)

Any element of S_{n+1} is expressed uniquely as a product of cyclic substitutions any two of which have disjoint letters. So, if we renumber $\{P_j\}_{j=0}^{n}$ suitably for each σ, Theorem 1.2 has the same expression that Theorem 1.1 has.

For each n dimensional simplex $P_0P_1P_2\cdots P_n$, the number of the equations given in Theorem 1.2 is equal to $d_{n+1} = \#\{\sigma \in S_{n+1} \mid \sigma(j) \neq j \text{ for any } 0 \le j \le n\}$. Namely the number of the generalizations of Theorem 1.1 type of Ceva's theorem on the triangle to simplex

$P_0P_1P_2\cdots P_n$ is equal to d_{n+1}. It is well known that $d_{n+1}=(n+1)!\left(\sum_{k=0}^{n+1}\frac{(-1)^k}{k!}\right)$.

Example. Let $n=3$. We have $d_4=9$. There are 9 equations of Theorem 1.1 type for any tetrahedron $P_0P_1P_2P_3$. The substitutions to consider are the following. As σ in Theorem 1.2, we have (0123), (0132), (0213), (0231), (0312), (0321), (01)(23), (02)(13), (03)(12).
The cases of (0123), (0132), (0213), (0231), (0312) and (0321) are treated in Theorem 1 in [1].
The cases of (01)(23), (02)(13) and (03)(12) are treated in Theorem 2 in [1].

One may say that there are only 9 generalizations of Ceva's theorem of the triangle to the tetrahedron.

Section 2. Theorem of another type for any integer $n>2$.

Let n be any integer>2. Let $P_0P_1P_2\cdots P_n$ be any n dimensional simplex with vertices $\{P_j\}_{j=0}^{n}$ in $\mathbf{R}^n$. Let T be any point in the interior of $P_0P_1P_2\cdots P_n$. Let b be any integer larger than 1. Let $\{U_j\}_{j=0}^{b}$ be such any sequence of points in $\mathbf{R}^n$ that $U_j\in\{P_x\mid 0\le x\le n\}$ for all the integers j with $0\le j\le b$, that $U_j\ne U_{j+1}$ for all the integers j with $0\le j\le b-1$ and that $U_0=U_b$. Then let $I(j,j+1)$ denote the intersection point of the line U_jU_{j+1} and (the hyper-plane $\supset(\{T\}\cup\{P_x\mid 0\le x\le n, P_x\ne U_j, P_x\ne U_{j+1}\})$) for each integer $j\in[0,b-1]$.
For arbitrary points P and Q in $\mathbf{R}^n$ we write $d(P,Q)=\left\|\overline{PQ}\right\|$. Notations being as above we obtain

Theorem 2.1. $\prod_{j=0}^{b-1}\frac{d(I(j,j+1),U_{j+1})}{d(U_j,I(j,j+1))}=1$.

Proof of Theorem 2.1. First we treat the case of $b=3$. Let π_0 be the minimum $\mathbf{R}$-linear submanifold in $\mathbf{R}^n$ containing T and all the points in $\{P_x \mid 0 \le x \le n, P_x \ne U_0, P_x \ne U_1, P_x \ne U_2\}$. Let π_0' be the minimum $\mathbf{R}$-linear submanifold in $\mathbf{R}^n$ containing T, U_0, U_1 and U_2. Note that the dimension of π_0 is $n-2$ and that the dimension of π_0' is 3. We have $\pi_0 \cap \pi_0'$ is a line (namely $\mathbf{R}$-linear submanifold of dimension 1). Let Q_0 be the intersection point of π_0 and the plane containing U_0, U_1 and U_2. Then the lines $U_0 I(1,2)$, $U_1 I(2,3)$ and $U_2 I(0,1)$ intersect at the point Q_0. So, we can apply Ceva's theorem to the triangle $U_0 U_1 U_2$. We have Theorem 2.1 for $b=3$.

Now assume Theorem 2.1 is true for b=some integer $k \ge 3$. Then we consider the case of $b=k+1$.

We have $U_{k+1} = U_0$. If $U_{k-1} = U_{k+1}$, this case resolves itself into the case of $b=k$ since $I(k-1,k) = I(k,k+1)$. So, we assume $U_{k-1} \ne U_{k+1}$. Let I denote the intersection point of the line $U_{k-1}U_{k+1}$ and (the hyper-plane$\supset (\{T\} \cup \{P_x \mid 0 \le x \le n, P_x \ne U_{k-1}, P_x \ne U_{k+1}\})$). By the case of $b=k$, we have $\dfrac{d(I, U_{k+1})}{d(U_{k-1}, I)} \prod_{j=0}^{k-2} \dfrac{d(I(j,j+1), U_{j+1})}{d(U_j, I(j,j+1))} = 1$. Let π be the minimum $\mathbf{R}$-linear submanifold in $\mathbf{R}^n$ containing T and all the points in $\{P_x \mid 0 \le x \le n, P_x \ne U_{k-1}, P_x \ne U_k, P_x \ne U_{k+1}\}$. Let π' be the minimum $\mathbf{R}$-linear submanifold in $\mathbf{R}^n$ containing T, U_{k-1}, U_k and U_{k+1}. Note that the dimension of π is $n-2$ and that the dimension of π' is 3. We have $\pi \cap \pi'$ is a line (namely $\mathbf{R}$-linear submanifold of dimension 1). Let Q be the intersection point of π and the plane containing U_{k-1}, U_k and U_{k+1}. Then the lines $U_{k-1} I(k,k+1)$, $U_k I$ and $U_{k+1} I(k-1,k)$ intersect at the point Q. Apply Ceva's theorem to the triangle $U_{k-1}U_kU_{k+1}$. We have

$$\frac{d(I, U_{k+1})}{d(U_{k-1}, I)} = \frac{d(I(k-1,k), U_k))}{d(U_{k-1}, I(k-1,k))} \cdot \frac{d(I(k,k+1), U_{k+1}))}{d(U_k, I(k,k+1))}.$$

Hence

$$\frac{d(I(k-1,k), U_k))}{d(U_{k-1}, I(k-1,k))} \cdot \frac{d(I(k,k+1), U_{k+1}))}{d(U_k, I(k,k+1))} \prod_{j=0}^{k-2} \frac{d(I(j,j+1), U_{j+1})}{d(U_j, I(j,j+1))} = 1.$$

Namely

$\prod_{j=0}^{k} \frac{d(\mathrm{I}(j,j+1),\mathrm{U}_{j+1})}{d(\mathrm{U}_j,\mathrm{I}(j,j+1))} = 1$. The case of $b = k+1$ is proven. Theorem 2.1 is proven.

(The case of $b = 2$ is trivial.)

Section 3. Case of $n = 4$.

The following theorem is different from Theorems 1.1 and 1.2 in this paper and Theorem 2 in [2]. We obtain

Theorem 3.1. *Let* $n = 4$. *One has*

$$\frac{m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4)m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3)}{m(\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4)} \cdot \frac{m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_3)m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)}{m(\mathrm{P}_1'\mathrm{P}_0\mathrm{P}_3\mathrm{P}_4)m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_4)} \cdot$$
$$\frac{m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_4)m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_1\mathrm{P}_4)}{m(\mathrm{P}_2'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_4)m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1)} \cdot \frac{m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1)m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2)}{m(\mathrm{P}_3'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2)m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2)} \cdot \frac{m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2)m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3)}{m(\mathrm{P}_4'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3)m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3)}$$
$$= 1.$$

Proof of Theorem 3.1. We use

If $a:b:c:d = t:u:v:w = w:x:y:z$, then $a:b:c:d = t-w:u-x:v-y:w-z$.

Using that and computing volumes of 4 dimensional simplexes we have

$$m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):m(\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)$$
$$=\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)$$
$$=\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)$$
$$=\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4).$$

By the same method,

$$m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_3):m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_4):m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4):m(\mathrm{P}_1'\mathrm{P}_0\mathrm{P}_3\mathrm{P}_4)$$
$$=\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_2\mathrm{P}_0\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_2\mathrm{P}_0\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_0\mathrm{P}_3\mathrm{P}_4),$$

$$m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1):m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_4):m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_1\mathrm{P}_4):m(\mathrm{P}_2'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_4)$$
$$=\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1):\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_3\mathrm{P}_0\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_3\mathrm{P}_1\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_0\mathrm{P}_1\mathrm{P}_4),$$

$$m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1):m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2):m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2):m(\mathrm{P}_3'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2)$$
$$=\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1):\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2):\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2):\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2),$$

and

$$m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2):m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3):m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3):m(\mathrm{P}_4'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3)$$
$$=\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2):\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3).$$

For simplicity, write

$A=\mu(\mathrm{TP_0P_2P_3P_4})$, $B=\mu(\mathrm{TP_1P_0P_3P_4})$, $C=\mu(\mathrm{TP_2P_0P_1P_4})$, $D=\mu(\mathrm{TP_3P_0P_1P_2})$ and $E=\mu(\mathrm{TP_4P_1P_2P_3})$.

Then the left side of the equation in Theorem 3.1 is equal to

$$\frac{CD}{AB}\cdot\frac{DE}{BC}\cdot\frac{AE}{CD}\cdot\frac{BA}{DE}\cdot\frac{CB}{EA}=\frac{A^2B^2C^2D^2E^2}{A^2B^2C^2D^2E^2}=1.$$

Theorem 3.1 is proven.

Since Theorem 3.1 holds true for any 4 dimensional simplex, we have

Corollary of Theorem 3.1. *Let* $\sigma=\begin{pmatrix}0 & 1 & 2 & 3 & 4\\ \sigma(0) & \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4)\end{pmatrix}$ *be any element of* S_5. *One can replace* P_j *by* $\mathrm{P}_{\sigma(j)}$ *for any* $0\le j\le 4$ *and* P'_j *by* $\mathrm{P}'_{\sigma(j)}$ *for any* $0\le j\le 4$ *in the equation of Theorem 3.1.*

Theorem 3.1 shows that Theorems 1.1 and 1.2 in Section 1 do not cover all possible generalizations of Ceva's theorem on the triangle to n dimensional simplexes with $n\ge 4$.

References

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