On the relative CR category

by
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Abstract

The purpose of the present note is to consider the relative CR category and to provide the set of morphisms with the functorial CR structures at least in an abelian CR category. For other CR categories, *e.g.,* complex one the relative notion is well known, but there seem not so many investigations in this direction to exist.

1   Introduction

Let us consider an arbitrary category A. We prepare some basic categorical notions necessary for the later arguments. First for surjectivity and injectivity, we are concerned in the category of sets. So it would be relevant to introduce the two different notions, set theoretic one and categorical one. We are indebted to [11] for the following definition:

Definition 1.1 (epic and monic)
(1)   A morphism $f : M \to N$ is *categorically surjective or epic*, if for any object X and any morphisms $g \neq h : N \to X$ we have $f^*(g) := g \circ f \neq h \circ f =: f^*(h)$. Namely the (contravariantly) induced morphism

$$f^* ; \mathrm{Hom}(N, X) \to \mathrm{Hom}(M, X)$$

is (set theoretically) injective.
(2)   A morphism $f : M \to N$ is *categorically injective or monic*, if for any object X and any morphisms $g \neq h : X \to M$ we have $f_*(g) := f \circ g \neq f \circ h =: f_*(h)$. Namely the (covariantly) induced morphism

$$f_* ; \mathrm{Hom}(X, M) \to \mathrm{Hom}(X, N)$$

is (set theoretically) injective.

It would be irrelevant to discuss what kind of implication holds concerning the two notions without any specific conditions on which it depends. In the following sections we will discuss this problem for specific categories. Let M be an object. Then a subobject L of M is a morphism $i : L \to M$ which is monic. As the dual notion we have a quotient object N of M defined by a morphism $p : M \to N$ which is epic. For specific categories of various geometric spaces, these notions differ from the geometric notions of subspaces or quotient ones of the original space. From a category $\mathcal{A}$ we can derive a new category $\mathcal{A}'$ in the following way. An object in $\mathcal{A}'$ is a triplet (M, f, N), where $f : M \to N$ is a morphism of an object M to another one N in $\mathcal{A}$. Then for two objetcs $f_1 : M_1 \to N_1$, $f_2 : M_2 \to N_2$, a morphism (g_1, h_1) of $(M_1, f_1, N_1) \to (M_2, f_2, N_2)$, is a pair of morphism $g_1 : M_1 \to M_2$ and $h_1 : N_1 \to N_2$, such that the following diagram commutes:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{f_1} & N_1 \\
\downarrow g_1 & & \downarrow h_1 \\
M_2 & \xrightarrow{f_2} & N_2.
\end{array}
$$

For later references we name this category $\mathcal{A}'$ the derived category of $\mathcal{A}$, and denote it always with the prime symbol ' attached to the original symbol, *e.g.*, $\mathcal{A}$. An object in this category is named derived object and a morphism is named derived morphism. Then it is clear that the derived category is well defined since the composed morphism of the morphism (g_1, h_1) with another one (g_2, h_2) of $f_2 : M_2 \to N_2$ to $f_3 : M_3 \to N_3$ is defined in the usual fashion; the following diagram is commutative.

$$
\begin{array}{ccc}
M_1 & \overset{f_1}{\to} & N_1 \\
\downarrow g_1 & & \downarrow h_1 \\
M_2 & \overset{f_2}{\to} & N_2 \\
\downarrow g_2 & & \downarrow h_2 \\
M_3 & \overset{f_3}{\to} & N_3.
\end{array}
$$

We denote the object MfN of a derived category sometimes by $f : M \to N$ following the expression in the original category. For a pair of objects MfN and $M'f'N'$, we denote the set of morphisms of MfN to $M'f'N$ by $\mathrm{Hom}(MfN, M'f'N')$.

2 Linear CR category revision

We review briefly some of the main results in [5] for the readers' convenience. First a CR vector space is a triplet $M = (U, R_U, J_U)$ of a real vector space U and a subspace R_U of U with the complex structure J_U on R. If the dimensions are concerned we have $m = \dim M$, $2r_U = \dim R_U$ and we refer $s_M := n - 2r_U$ as CR-codimension of the object M. The pair (R_U, J_U) is referred to the CR-structure of the object M and (r_U, s_M) is sometimes called the type of the CR-structure of M. For simplicity of notation the suffixes $_{U,M}$ are omitted if there is no fear of confusion. The CR structure is *totally real* iff $r = 0$ and *totally complex* iff $s = 0$. Let $N = (V, R_V, J_V)$ be another CR vector space with the CR structure of type (r_V, s_N). Then a *(CR)-morphism* $f : M \to N$ is an **R**-linear mapping compatible with the respective CR structures:

$$ f(R_U) \subset R_V $$

$$ f(J_U(u)) = J_V(f(u)), \quad u \in R_U $$

Before proceeding further, letting $M = (U, R_U, J_U)$ be a CR vector space, we have to prepare the following

Definition 2.1 (*subspace*)   A CR space $L = (T, R_T, J_T)$ is a (geometric) CR subspace of M if
(i) L is an **R**-subspace of U,
(ii) R_T is an J_U invariant subspace of R_U, and J_T is the restriction of J_U to R_T.

Thus the natural inclusion $\iota : L \hookrightarrow M$, that is injective, is monic (see the Theorem 2.1 below), and hence defines a subobject in the sense of the 1st section. There exists a more strong subspace notion, which is the *induced subspace* as was mentioned in [5]:

Definition 2.2 (*induced CR structure*)   A CR subspace $L = (T, R_T, J_T)$ of M is a *CR subspace with the induced CR structure* (R_T, J_T), or an *induced CR subspace* for short, if
(iii) (R_T, J_T) is the CR structure induced from R_U; $R_T = (T \cap R_U) \cap J_U(T \cap R_U)$
or equivalently
(iii)' R_T is the maximal J_U invariant **R**-subspace of T.

The induced CR structure is characterized by the following universality condition.

Theorem 2.1 (cf. [5]) *Let T be an $\mathbf{R}$-subspace of a CR vector space M. Then the natural inclusion $\iota : L \hookrightarrow M$ is a morphism, where $L = (T, R_T, J_T)$ is the induced CR subspace of M. The CR structure (R_T, J_T) is maximal in the set of CR structures on T with this property.*

Proof. Let (R, J) be any other CR structure on T such that ι is a morphism. Then we have by definition $R = \iota(R) \subset R_U$. Since R is a (J invariant) subset of T we have $R \subset T \cap R_U$ and $J(R) \subset J(T \cap R_U)$, which deduce $R \subset (T \cap R_U) \cap J(T \cap R_U)$.   Q.E.D.

Definition 2.3 (*generic*)   An $\mathbf{R}$-subspace G of U is *generic* in $M = (U, R, J)$, if

$$U = G + J(R \cap G).$$

Then we have

$$R = R \cap G + J(R \cap G),$$

which implies G and J generate the CR vector space $U = (U, R, J)$.

Definition 2.4 (*generically surjective*)   An $\mathbf{R}$-linear morphism $f : M \to N$ is *generically surjective* if the *set theoretic image* $f(M)$ is generic in the target space N.

As applications of the above definition, we have the following definition and theorems.

Definition 2.5 (*complexification* [5])   Let U be a CR vector space. Then a complexification of U is a pair (f, V) of a monomorphism $f : U \to V$ of U into a totally complex CR vector space V satisfying the following universality condition:
(uc)   For any totally complex object W and a morphism $g : U \to W$ in $\mathcal{LCR}$, there exists uniquely a morphism $h : V \to W$, such that the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
 & \searrow g & \downarrow h \\
 & & W
\end{array}
$$

Then a complexification of U, denoted $(f, U^{\mathbf{C}})$, does exist uniquely in the sense as follows :

Theorem 2.2   *The complexification $(f, U^{\mathbf{C}})$ exists uniquely up to isomorphisms; if (f', V') is another complexification of U, then we have an isomorphism $\iota : V \to V'$ such that the following diagram commutes.*

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
 & \searrow f' & \downarrow \iota \\
 & & V'
\end{array}
$$

Furthermore the complexification f is realized as a generic (CR) imbedding $f : U \to U^{\mathbf{C}}$ of U into a totally complex CR vector space $U^{\mathbf{C}}$ (see [5] for the definition and more details of the generic embeddings).

Proof. The definition above is a little bit different from the original one in [5], and the theorem itself is formulated in a different way, but we may follow the main idea of proof there. The uniqueness follows from the (uc). We may drop the monic condition of f, since it seems superfluous, but yet to be proved without this assumption.   Q.E.D.

Theorem 2.3   *In the $\mathcal{LCR}$ a morphism g is*
(1) *epic iff it is generically surjective,*
(2) *monic iff it is set-theoretically injective.*

3 Homomorphisms in the linear CR category

Let $M = (U, R_U, J_U)$, $N = (V, R_V, J_V)$ be CR vector spaces. Now we consider the set of CR morphisms, denoted $\mathrm{Hom}(M, N)$, as CR object. It has the natural $\mathbf{R}$-module structure, which is compatible with the CR structure on the respective objects M, N. As in [5] we can equip $H := \mathrm{Hom}(M, N)$ with the canonical CR structure (R_H, J_H) in the following manner.

$$R_H := \{f \in H | f(U) \subset R_V\}$$

$$(J_H(f))(u) := J_V(f(u)), \quad u \in U$$

Then $H = (H, R_H, J_H)$, thus obtained, is a CR vector space with the maximal CR structure, in the sense it is maximal in the set of CR structures on H such that the natural operation ρ of H on the source object M

$$\rho : H \times M \to N,$$

is CR. More exactly speaking the mapping ρ is *CR bilinear* and is referred to as evaluation mapping in [5]. The maximality was not mentioned explicitly there. Here we only remark the maximality of R_H without any proof. The exact statement (with proof) of this fact is essentially given in a similar theorem in the 5th section. For the special case of $M = N$, we adopt the conventional notation and terminology

$$E := \mathrm{End}(M)$$

in stead of $\mathrm{Hom}(M, M)$. Then E has the multiplication μ defined by the composition of morphisms:

$$\mu : E \times E \to E,$$

which is CR bilinear. This multiplication satisfies the associativity and distributivity, respectively and hence defines the (non commutative CR algebra structure on E (see [6] for the definition of CR algebra). The invertible elements of E, denoted $G := \mathrm{GL}_{CR}(M)$, constitute the *general linear CR group of M*, which is a real Lie subgroup of the general linear group $\mathrm{GL}(U, \mathbf{R})$ of the real vector space U. The group G has the invariant CR structure, in the sense that the restriction $\rho|G$ of ρ to G:

$$\rho|G : G \times M \to M$$

is a CR morphism in the (non linear) CR category (see the next section and [12]). In this sense we referred to this action as a CR Lie group action on a CR manifold M [12, 13, 14]. In the following sections we investigate a little bit more general CR category, which is non-linear. In this extended category the isomorphism group of the object M, denoted $A := \mathrm{Aut}(M)$, contains $G := \mathrm{GL}_{CR}(M)$ as an abstract subgroup. If A has an invariant CR structure, compatible with that of G, then we may have the CR inclusion (which is yet to be defined precisely)

$$\iota : G \hookrightarrow A.$$

4 Affine CR category

In this section we consider a more general CR category than that of CR vector spaces. Take a smooth m-dimensional manifold M with the tangent bundle TM. Here "smooth" means the $\mathcal{C}^\infty$ differentiability. A smooth CR manifold $M = (M, R_M, J_M)$ is a triplet of smooth manifold M and an $\mathbf{R}$-subbundle R_M of TM $\mathbf{R}$-rank $2r$, and an almost complex structure tensor J_M on R_M. We refer to the pair (R_M, J_M) as the CR structure of M, the pair (r, s) as the type

and s as the CR codimension of the CR structure, respectively, where $s = m - 2r$. As is the linear case, the CR structure is *totally real* iff $r = 0$ and *totally complex* if iff $s = 0$. Then a morphism $f : M \to N$ of M to another CR manifold $N = (N, R_N, J_N)$ is a smooth mapping compatible with the respective CR structures of M, N:

$$df(R_M) \subset R_N$$
$$J_N((df)(u)) := df(J_M(u)), \quad u \in R_M,$$

where df denotes the differential of f. Thus the category of CR manifolds is defined and denoted $\mathcal{CR}$. Then $\mathcal{LCR}$ can be considered as a subcategory of $\mathcal{CR}$ by assigning to each CR vector space $U = (U, R_U, J_U)$ the associated smooth CR-manifold denoted $M = (M, R_M, J_M)$, where M denotes the smooth manifold defined by the real vector space U with the natural smooth structure $U = \mathbf{R}^m$ under an arbitrary base $\{e_1, e_2, \ldots, e_m\}$ and R_M denotes the subbundle of $TM = U \times \mathbf{R}^m$ defined at each point $p \in M = U$ by $R_p M = \mathbf{R}^{2r}$, which is identified with R_U irrespective of the point p involved. A morphism $f : U \to V$ in $\mathcal{LCR}$ naturally defines a smooth (linear) CR mapping of M to $N = (N, R_N, J_N)$, where N is defined by CR vector space $V = (V, R_V, J_V)$ as in the case of M, since its differential $df : TM \to TN$ is represented by a constant matrix, which is CR at each point $p \in M$ (see [13] for matrix representations of linear CR mappings). Usually we impose an additional assumption *integrability* on the CR structure (RM, JM) defined by

$$[JX, JY] + J[X, JY] + J[JX, Y] = [X, Y], X, Y \in \Gamma(RM),$$

where $\Gamma(RM)$ denotes the (germs of local) smooth sections of RM. This is equivalent to the complexified condition [1, 3, 5]:

$$[\Gamma(AM), \Gamma(AM)] \subset \Gamma(AM)$$

the so-called involutiveness of AM, where AM denotes the anti-holomorphic part of the complexified bundle $R_M^{\mathbf{C}} := R_M \otimes \mathbf{C}$ of R_M and $\Gamma(AM)$ denotes its sections. As is well known the integrability condition is satisfied if the CR structure is induced from an imbedding of M into a totally complex CR manifold. For our linear case the CR vector space $U = (U, R_U, J_U)$ admits a complexification $U^{\mathbf{C}}$ ([5]), which is a totally complex affine CR space by definition. We can verify the CR structure of U coincides with the induced structure and hence integrable. We can deduce the integrability directly by calculating the bracket relation above, since the tangent bundle TU is trivial.

Remark 4.1 We remark that $\mathcal{LCR}$ is not a *full subcategory* of $\mathcal{CR}$, since the smooth CR mappings are far more abundant than the linear ones. One of the most important reasons why we have to consider more general CR category than the linear one is that a CR vector space V is not an abelian CR group in this category, since the addition is not a CR morphism even factor-wise separately. This problem can be solved if we consider the general CR category, which is however too general to obtain more specific results. So now it's time to consider a tractable and moderately general subcategory which contains $\mathcal{LCR}$.

Definition 4.1 (*Affine CR space*) An *affine CR space* is a triplet (A, R_A, J_A) of an affine $\mathbf{R}$-space A (of dim $A = n$), a rank 2r subbundle of the tangent bundle TA of A and the almost complex structure tensor J_A of R_A. We refer to the pair (R_A, J_A) as the CR structure of the affine CR space, and (r, s) as the type of the CR structure. Let $B = (B, R_B, J_B)$ be another affine space. Then an affine mapping $f : A \to B$ is an *affine CR morphism*, if its differential $df : T_A \to T_B$ is compatible with the respective CR structures:

$$df(R_A) \subset R_B$$

$$df(J_A(u)) = J_B(df(u)), \quad u \in R_A,$$

as a bundle mapping.

Thus the category of affine CR spaces, denoted $\mathcal{CRA}$, is defined.

Remark 4.2 It is more appropriate to define $\mathcal{CRA}$ not relying on the tangent bundle structure, since it is trivial as in the linear case. But for the present moment we adopt the above definition so as to imbed $\mathcal{CRA}$ in $\mathcal{CR}$, although it is not a full subcategory as is the case in the linear one.

We can easily observe this category is closed under direct product with the self-evident product CR structure. Namely we have the following

Definition 4.2 (Direct product) For objects A, B in $\mathcal{CRA}$, we have the product affine space, denoted $A \times B$, defined by
$$(A \times B, R_A \oplus R_B, J_A \oplus J_B),$$

where $A \times B$ denotes the set theoretic direct product of $\mathbf{R}$- affine spaces A, B and $R_A \oplus R_B$ the direct sum of the respective tangent subbundles and $J_A \oplus J_B$ the direct sum of the respective tensor fields. The CR structure $(R_A \oplus R_B, J_A \oplus J_B)$ is referred to as the product CR structure.

Exactly speaking we need a proof of the above theorem, which is, however, left to the readers since it is not difficult but is only tedious. Then in the situation as above we have the following

Theorem 4.1 *The respective projections $p_A : A \times B \to A$, $p_B : A \times B \to B$ are CR are morphisms in $\mathcal{CRA}$. Furthermore the product CR structure is characterized as the maximal CR structure such that they are morphisms.*

We have already pointed out that a CR vector space U can be considered a CR affine space A. If there is no fear of confusion we may use the same notation U in stead of A. Anyway U has an additive group structure. Then we may introduce the notion of an affine CR group as follows:

Theorem 4.2 *A CR vector space$U$, considered canonically an CR affine space, is an* affine CR group *in the sense that the group (additive) multiplication or addition*

$$\mu : U \times U \to U$$

defined by $\mu(u,v) = u + v, uv \in U$ is an affine CR morphism with respect to the product affine structurein $U \times V$.

Proof. We have given the main points of the theorem in the process of defining $\mathcal{CRA}$. Q.E.D.

5   Relative linear CR category

In the 1st section we have considered the derived category in the most general settings, with no specific assumptions either on the object or on the morphism. Here in this section we confine ourselves to the case where the target original object N of the derived object $f : M \to N$ is a fixed object, say, B independent of the source object M, but the morphism f varies in accordance with the source M. In the usual categorical arguments this object is the so called B-object and denoted simply (M, f), and a morphism $(g, h) : (M, f) \to (M', f')$ is the so called B-morphism if h is an identity morphism of B. Henceforth we refer to the fixed

object B as *base object* and denote the morphism (g, id_B) simply by g following the usual convention. The derived category thus confined is named the relative category of the original one. The argument of this section is partly based on the theory developed in [5]. Let $\mathcal{CR}$ be the category of CR manifolds, or (general) CR category (may be with CR singularities). Then the relative category, $\mathcal{CR}'$, or relative CR category, is defined following the procedures in the preceding section. Confining ourselves to the linear case we have the category $\mathcal{LCR}$ of CR vector spaces, or linear CR category. Then the relative category $\mathcal{LCR}'$, or relative linear CR category, is defined. Now we examine some properties of the relative linear CR category. Before stating our main result we give

Remark 5.1   An object M in the B-object (M, f) has a CR structure (R_M, J_M) in the original $\mathcal{LCR}$. We use this CR structure as that of (M, f) in $\mathcal{CR}'$. Thus we can call $\mathcal{LCR}'$ the relative CR category or CR relative category either.

Now we consider two B-objects $M = (M, f), N = (N, g)$ in $\mathcal{LCR}'$ where B is a fixed base object in $\mathcal{LCR}$. We denote the set of B-morphisms of M to N by $H_B := \mathrm{Hom}_B(M, N)$, that is a subset of $H = \mathrm{Hom}(M, N)$ as was mentioned in the 3rd section. Then following the procedures in the previous section, the CR vector space H can be considered a CR manifold $H = (H, R_H, J_H)$, whose dimension h and the type (p, q) of the CR structure (R_H, J_H) are determined in [5] (Lemma 2.7). The tangent bundle of H_B, denoted TH_B, is a subbundle of TH, since H_B is a smooth submanifold of H. Then we have the induced CR structure $R_h(H_B)$ on each tangent space $T_h H_B$ at $h \in H_B$ (see the Definition 2.1 and 2.2 in the 2nd section for subspace and induced CR structure). Then we denote the subset $\cup_{h \in H_B} R_h(H_B)$ of TH by R_{H_B}. Let J_{H_B} denote the tensor field defined by $(J_h(H_B)(dk))(u) = J(dk(u))$, where dk is a tangent vector at h, with $k \in H_B$ and dk is a differential of k. Then we have the following

Theorem 5.1 *The set of B-morphisms of M to N, denoted $H_B := \mathrm{Hom}_B(M, N)$, is an affine CR space with the canonical CR structure (R_{H_B}, J_{H_B}) such that the natural operation ρ*

$$\rho : \mathrm{Hom}_B(M, N) \times (M, f) \to (N, g)$$

is CR. Further it is maximal in the set of CR structures satisfying the above condition (see the 2nd section and [5] for details of dominance relation of CR structures).

Proof.   We give only a brief sketch of the main points. Let us take an arbitrary B-morphism, say, $h : (M, f) \to (N, g)$. Then it satisfies the commutativity relation $f = g \circ h$. It is easily observed that this condition is a non homogeneous linear equation $E = E(f, g; h) = 0$ with respect to the unknown variable h, where f and g are fixed parameters. Since h is an element of the CR vector space $H = \mathrm{Hom}(M, N)$, the set of solutions of the non-homogeneous linear equation $E = 0$ forms an *affine subspace* of the total space H. It is proved that R_{H_B} is an **R**-subbundle of the tangent bundle TH_B of H_B. The key point is the equidimensionality of each CR structure $R_h(H_B)$ independent of the morphism h. Further we can show that the almost complex structure tensor $J_h(H_B)$ preserves the **R**-subspace $R_h(H_B)$. These phenomena are based on the fact that each CR structure on each morphism h is invariant through translations by any vectors in U, V.   Q.E.D.

For a detailed proof, readers are referred to [22] which may be published somewhere.

6   Concluding remarks

We have examined the structure of the set of morphisms in the relative CR category. We can prove some complexification theorems in $\mathcal{CRA}'$, which are however, not stated here. We

may give them in explicit forms in the forthcoming [22]. We would like to know more in the case of $M = N$; $E_B =_B (M)$ has the affine CR algebra structure $(R_{E_B}, J_{E_B}$. It is non-abelian in general and so we have the (non-trivial) associated affine CR Lie algebra $\mathcal{GL}_{CR}(E_B)$. Our farther target is the similar problem in a more general setting. There is a hope that some affirmative answer is obtained in a relative non-linear CR category of restricted type for which we refer the readers to [22]

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