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Generalization of Ceva's theorem to the n dimensional simplex

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Abstract. We generalize Ceva's theorem to the n-dimensional simplex.

Let n be any integer>2. Let $\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n$ denote an arbitrary n dimensional simplex in Euclidean space $\mathbf{R}^n$, whose vertices are $\{\mathrm{P}_j|\ 0\le j\le n\}$. Let μ denote the standard Euclidean measure (=volume) on $\mathbf{R}^n$. Let m denote the standard Euclidean measure on $\mathbf{R}^{n-1}$. Let T be any point in the interior of $\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n$. For each integer j with $0\le j\le n$, write $C_j=\{\mathrm{P}_x\mid 0\le x\le n, x\ne j\}$. For each integer j with $0\le j\le n-1$, write $D_j=C_j\bigcap C_{j+1}$. Write $D_n=C_n\bigcap C_0$. For each integer j with $0\le j\le n$, let H_j denote the $n-1$ dimensional hyper-plane containing all the points in C_j. For each integer j with $0\le j\le n$, let P'_j denote the point of the intersection of line $\mathrm{P}_j\mathrm{T}$ and the hyper-plane H_j. For arbitarary finite points $\{\mathrm{V}_j\mid 1\le j\le k\}$ in Euclidean space $\mathbf{R}^n$, let $<\{\mathrm{V}_j\mid 1\le j\le k\}>$ denote the minimum closed convex subset, of $\mathbf{R}^n$, containing all the points $\{\mathrm{V}_j\mid 1\le j\le k\}$.

Section 1. Case of $n=4$.

We studied the case of $n=3$ in my article "Generalization of Ceva's theorem on the triangle to the tetrahedron" in Science Reports of the Faculty of Education Gifu University (Natural Science), Vol. 31, (February 2007), pp.7-9 already.

We obtain

Theorem 1. *Let* $n=4$. *One has*

$$\frac{m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4)m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4)m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3)}{m(\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)^3}\cdot\frac{m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_3)m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_4)m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)}{m(\mathrm{P}_1'\mathrm{P}_0\mathrm{P}_3\mathrm{P}_4)^3}\cdot$$
$$\frac{m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1)m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_4)m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_1\mathrm{P}_4)}{m(\mathrm{P}_2'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_4)^3}\cdot\frac{m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1)m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2)m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2)}{m(\mathrm{P}_3'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2)^3}\cdot$$
$$\frac{m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2)m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3)m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3)}{m(\mathrm{P}_4'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3)^3}$$
$$=1$$

Proof of Theorem 1. We use

If $a:b:c:d=t:u:v:w=s:x:y:z$, then $a:b:c:d=t-s:u-x:v-y:w-z$.

Computing volumes of 4 dimensional simplexes and using that we have

$$\begin{aligned}
&m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):m(\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):m(\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)\\
&=\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{P}_0\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)\\
&=\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_0'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)\\
&=\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4).
\end{aligned}$$

By the same method,

$$\begin{aligned}
&m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_3):m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_0\mathrm{P}_4):m(\mathrm{P}_1'\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4):m(\mathrm{P}_1'\mathrm{P}_0\mathrm{P}_3\mathrm{P}_4)\\
&=\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_2\mathrm{P}_0\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_2\mathrm{P}_0\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_0\mathrm{P}_3\mathrm{P}_4),
\end{aligned}$$

$$\begin{aligned}
&m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1):m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_0\mathrm{P}_4):m(\mathrm{P}_2'\mathrm{P}_3\mathrm{P}_1\mathrm{P}_4):m(\mathrm{P}_2'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_4)\\
&=\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1):\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_3\mathrm{P}_0\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_3\mathrm{P}_1\mathrm{P}_4):\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_0\mathrm{P}_1\mathrm{P}_4),
\end{aligned}$$

$$\begin{aligned}
&m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1):m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2):m(\mathrm{P}_3'\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2):m(\mathrm{P}_3'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2)\\
&=\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1):\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2):\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2):\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2),
\end{aligned}$$

and

$$\begin{aligned}
&m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2):m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3):m(\mathrm{P}_4'\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3):m(\mathrm{P}_4'\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3)\\
&=\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2):\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_0\mathrm{P}_1\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3):\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3).
\end{aligned}$$

For simplicity, write

$A=\mu(\mathrm{T}\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4)$, $B=\mu(\mathrm{T}\mathrm{P}_1\mathrm{P}_0\mathrm{P}_3\mathrm{P}_4)$, $C=\mu(\mathrm{T}\mathrm{P}_2\mathrm{P}_0\mathrm{P}_1\mathrm{P}_4)$, $D=\mu(\mathrm{T}\mathrm{P}_3\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2)$ and $E=\mu(\mathrm{T}\mathrm{P}_4\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3)$.

Then the left side of the equation in Theorem 1 is equal to

$$\frac{BCD}{A^3}\cdot\frac{DCE}{B^3}\cdot\frac{DAE}{C^3}\cdot\frac{BAE}{D^3}\cdot\frac{CBA}{E^3}=\frac{A^3B^3C^3D^3E^3}{A^3B^3C^3D^3E^3}=1.$$

Theorem 1 is proved.

Since Theorem 1 holds true for any 4 dimensional simplex, we have

Corollary 1 of Theorem 1. *Let* $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \sigma(0) & \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{pmatrix}$ *be any element of* S_5. *One can replace* P_j by $\mathrm{P}_{\sigma(j)}$ for any $0 \le j \le 4$ *and* P'_j by $\mathrm{P}'_{\sigma(j)}$ for any $0 \le j \le 4$ *in the equation of Theorem 1.*

Section 2. Case of any integer $n \ge 3$.

Let *j* be any integer with $0 \le j \le n$. Write $Q_j = \{\mathrm{P}_x \mid 0 \le x \le n, x \ne j\}$. Write $\Lambda_j = \{S \mid S \subset Q_j, |S| = n-1, S \ne D_j\}$. Then we have $|\Lambda_j| = n-1$. We obtain

Theorem 2. $$\prod_{j=0}^{n} \left(\frac{\prod_{\xi \in \Lambda_j} m(<\{\mathrm{P}'_j\} \cup \xi >)}{m(<\{\mathrm{P}'_j\} \cup D_j >)^{n-1}} \right) = 1.$$

Proof of Theorem 2. We consider Theorem 2 for the order $(\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2, \cdots, \mathrm{P}_n, \mathrm{P}_0)$.

By the same way as in the proof of Theorem 1 we have
$$\frac{\prod_{\xi \in \Lambda_j} m(<\{\mathrm{P}'_j\} \cup \xi >)}{m(<\{\mathrm{P}'_j\} \cup D_j >)^{n-1}} = \frac{\prod_{\xi \in \Lambda_j} \mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup \xi >)}{\mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup D_j >)^{n-1}} .$$

We obtain $\prod_{j=0}^{n} (\prod_{\xi \in \Lambda_j} \mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup \xi >)) = \prod_{j=0}^{n} \mu(<\{\mathrm{T}, \mathrm{P}_j\} \cup D_j >)^{n-1}$.

Theorem 2 is proved.

Corollary of Theorem 2. *Let* $\tau = \begin{pmatrix} 0 & 1 & 2 & \cdots & n \\ \tau(0) & \tau(1) & \tau(2) & \cdots & \tau(n) \end{pmatrix}$ *be any element of* S_{n+1}. *Rewrite* P_j *for* $\mathrm{P}_{\tau(j)}$ *with all* $0 \le j \le n$ *and* P'_j *for* $\mathrm{P}'_{\tau(j)}$ *with all* $0 \le j \le n$. *Then apply Theorem 2. Then we have the equation for simplex*

$P_{\tau(0)}P_{\tau(1)}P_{\tau(2)}\cdots P_{\tau(n)}$.

(We can consider Theorem 2 for any order $(P_{\tau(0)},P_{\tau(1)},P_{\tau(2)},\cdots,P_{\tau(n)},P_{\tau(0)})$.)