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Some remarks on automorphism groups of toroidal/(H,C) groups

Dedicated to Professor Akihiko Morimoto on his 80 th birthday,
by

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Abstract

The purpose of the present note is to discuss the automorphisms of an arbitrary (H,C) space. For the case of group manifold, (H,C) space is toroidal if we follow the definition/naming of toroidal group by Vogt [15] (see also Abe-Kopfermann [1] and Kazama-Umeno [3]); if a complex Lie group G has no non-constant (global) holomorphic functions, it is said to be a *toroidal group*, which is first defined and named (H,C) group by Morimoto [10] (also see [11] by the present author) in the more general setting. As a generalization of compact complex torus, toroidal group G is generically isomorphic with holomorphic automorphism group Aut(G) up to connectivity. We consider the problem related to this result. [1]

1 Introduction

First we give a short summary of fundamentals of complex Lie groups for the readers' convenience. A group G is a complex Lie group if it is a complex manifold whose group operation $\mu : G \times G \to G$, defined by $\mu(g,h) := gh^{-1}$, is holomorphic. For the present moment we assume G to be connected unless otherwise stated. As was stated in [11](see also [7, 8, 10]) we have the two fundamental structure theorems of complex Lie group G as follows:

Theorem 1.1 *Let K be a maximal compact subgroup of G. Then the complexification K^c of K (see [7] for details) is a closed reductive complex subgroup G and the factor space G/K^c is biholomorphic to a complex affine space $\mathbf{C}^\alpha$. Hence G is biholomorphic to the product complex manifold $K^c \times \mathbf{C}^\alpha$ (see [11] for the precise definition of the complex characteristic index $\alpha = \alpha(G)$ of G). If G itself is a reductive complex Lie group then G contains a unique maximal Stein subgroup S, such that S is normal and the factor group G/S is a complex torus. Thus G has the structure of holomorphic fiber bundle over complex torus of dimension $\beta = \beta(G)$ with Stein fiber $S \times \mathbf{C}^\alpha$.*

Theorem 1.2 *There is a maximal toroidal subgroup T of G, contained in the center $Z = Z(G)$ of G, such that the factor group G/T is a Stein group. Then T is equal to the Zariski closure (in G) of the maximal complex subgroup K_o of K. The complex*

[1] Key Words and Phrases: no non-constant holomorphic functions, complex Lie group, automorphism group, (H,C) group, toroidal group

analytic invariant index $\beta = \beta(G)$ is defined to be the complex dimension $\delta(K_o)$ of K_o and is equal to the completeness *of G, denoted $com(G)$ which is independent of the choice of K [11].*

The readers are referred to [10] for a detailed proof of the second theorem and we instead give some basic characteristics of S, K_o and T, since it would be useful for them to understand our main result. A complex Lie group G is a Stein group if it is a Stein manifold as a group manifold. A subgroup S of an arbitrary complex Lie group G is a maximal Stein subgroup if it is a Stein group, and there are no larger Stein subgroups containing S. As for the second theorem we should mention that T is defined as follows. Let X be a complex space, and let $\mathcal{O}(X)$ be the set of global holomorphic functions on X and we set $X_x := \{x \in X; f(z) = f(x), f \in \mathcal{O}(X)\}$. Then it is observed that X_x is a complex analytic subvariety which is invariant under the biholomorphic automorphism group $Aut(X)$ of X [11]. Morimoto [**?**] proved $T = G_e$ is a central complex subgroup of G and the quotient complex Lie group G/T is a Stein group, such that $\mathcal{O}(G) = \mathcal{O}(G/T)$. Thus as a corollary to the above theorem we find $\mathcal{O}(G)$ is a Stein algebra for every complex Lie group G.

2 The automorphism group of a complex space

We can formulate some problems in the holomorphic symmetry of a complex space X. First we consider the set of all biholomorphic self-mappings of X, which forms a group, called the *automorphism group* of X, denoted $Aut(X)$, with the multiplication defined by the composition of two automorphisms. Then it would be natural to ask the following

Question What is $Aut(X)$? Is it a Lie transformation group (with the compact open topology) which acts holomorphically on X ? If so, determine (the dimension of) $Aut(X)$, *e.g.*,, its group structure.

In some well known cases this task has been done by several authors (see [4] for classical results as a bibliography). There are answers in the most general setting of complex space X. The one is for the bonded domain case, which says $Aut(X)$ is a totally real CR Lie group. Another one is for the compact case, which says $Aut(X)$ is a totally complex CR Lie group with the invariant complex structure J, in the sense that the canonical action $Aut(X) \times X \to X$ is holomorphic with respect to J. In these two cases some concrete results are obtained for specific objects X. As far as the present author knows there are few results beside these cases even in the CR category [12, 13]. For compact cases there is a general result that compact complex surface X of general type doesn't admit any non-trivial automorphisms. We may roughly say that X admits abundant automorphisms if X is akin to homogeneous. Precisely saying we should give the following

Definition 2.1 A complex space X is (holomorphically) *heterogeneous* if X admits no non-trivial holomorphic automorphisms.

The opposite extreme is the well known

Definition 2.2 A complex space X is *homogeneous* if $Aut(X)$ acts transitively on X.

There are no definitions for the intermediate cases. Before giving the intermediate definitions we prepare some preliminary notions. Take a point $x \in X$. We consider the orbit O_x of x by $Aut(X)$; $O_x = \{gx | g \in |Aut(X)\}$. In general it is not known whether O_x is a complex analytic subvariety or not, except the special cases defined by some additional assumptions. Among others, we should mention the case where $Aut(X)$ is the Lie transformation group G acting smoothly on X. Then we can smoothly identify the subspace O_x with the coset space G/G_x equipped with the quotient topology, where G_x is the isotropy subgroup of G at x. In general G is a CR Lie transformation group with the invariant CR structure and we can consider every orbit O_x to be a CR submanifold with the induced CR structure [12, 13, 14]. In this direction we have no results on the CR property(*e.g.*,, CR codimension *etc.*,) of the orbits for specific examples. To treat more general cases we consider the Zariski closure $\overline{O}_x$ of each orbit O_x in X. For simplicity we denote it by Z_x, which is a complex analytic subvariety by definition.
We define the *(local complex) homogeneity* h_x at a point $x \in X$ by the complex dimension of Z_x.

$$h_x := \dim_{\mathbf{C}} Z_x$$

Then we define the *(global complex) homogeneity* $h(X)$ of X by the maximum of h_x.

$$h(X) := \max\{h_x | x \in X\}$$

Now we should give the following

Definition 2.3 A complex space X is of homogeneity h or X is homogeneous of degree h iff $h(X)$ is equal to h.

Thus in this terminology if X is homogeneous then obviously $\delta(X) = h(X)$, where $\delta(X)$ denotes the complex dimension of X. But the converse is not true in general. Some people say X is almost homogeneous if there exists an open orbit O_x which is (topologically) dense in X. In this article we adopt the terminology that X is *quasi-homogeneous* (with respect to $Aut(X)$), if X is homogeneous of degree $\delta = \delta(X)$; $h(X) = \delta(X)$. This is equivalent to the existence of a Zariski dense orbit O_x, which is sometimes expressed a generic orbit.

Remark 2.1 It should be mentioned even if a generic orbit exists, it is not unique in general as will be shown in the next section.

The category of complex spaces is defined naturally; a morphism of two objects $X \to Y$ is a holomorphic mapping. Then taking the holomorphic function algebra $O(X)$ we obtain the contravariant functor $\mathcal{F}$ to the category of complex commutative algebras with the usual morphisms. For those spaces of high homogeneity we could get lots of information from their function algebras. For an automorphism $f : X \to Y$ we have the induced automorphism $f^* : O(Y) \to O(X)$; $f^* = \mathcal{F}(f)$, and $O(X) = \mathcal{F}(X)$,*etc.*,. Especially for Stein object X, this functor is contravariantly isomorphic and we can treat the homogeneity equivalently in both categories. But in the (H,C) cases this functor cannot play any role, since it is a trivial functor. But there are infinitely many (H,C) spaces which are homogeneous. So for this class we should directly attack the spaces themselves and their automorphisms. We will try this task in the next section.

3 (H,C) space

In the previous section we gave an idea to classify complex spaces by the *homogeneity*. Here in the present section we would like to classify complex spaces by the abundance of its holomorphic functions $O(X)$. The most abundant one is the class of Stein spaces, and the opposite extreme is the class of (H,C) spaces due to Morimoto [**?**]. The compactness of X is a sufficient condition to belong to (H,C) class. By Hartogs phenomena it is not a necessary condition, since a Zariski open subset Y of a compact complex space X belongs to (H,C) class, if the complementary subset $X - Y$ is an analytic subvariety of codimension greater than one. In order to simplify the expression from now on we use the term (H,C) not only as an attributive but also a predicate adjective; instead of saying X belongs to (H,C) class we say X is (H,C). So we are mainly interested in *holomorphically non-compact* cases; if Y has a compactification X such that complex codimension of $(X - Y)$ is less than 2, Y is said to be *holomorphically compact*. In order to systematically study (H,C) spaces it would be convenient to introduce the following

Definition 3.1 An (H,C) space X is reducible if there exist positive dimensional closed subspaces X_1, X_2 such that X is biholomorphically equivalent to the product space $X_1 \times X_2$. If it is not reducible then it is said to be irreducible. If X is expressed as a product space $\Pi_{i=1}^{k} X_i$ of irreducible subspaces $\{X_i | i = 1.2, \dots k\}$ we say each X_i is an irreducible component of X.

Remark 3.1 Each irreducible component is obviously (H,C) and hence may be called irreducible (H,C) component or factor.

Remark 3.2 At the present moment we have no general results on the uniqueness of these components up to order.

Remark 3.3 Abe [1] proved a quasi-abelian variety is a quasi-projective variety, which suggests the similar philosophy as ours in the (H,C) case, since both object have a projective algebraic compactification.

As for an (H,C) space X in general we will develop systematic approach not here but in the forthcoming paper [14]. Instead we treat an (H,C) complex Lie group G in the next section. Owing to the pioneer work of Morimoto [**?**] it is an abelian group whose universal covering group $\tilde{G}$ is a complex vector space$\mathbf{C}^n$. We can take $\mathbf{R}$ linearly independent vectors $\{e_1, e_2, \dots, e_n\}$ so that they are $\mathbf{C}$ linearly independent, hence form a base of complex vector space $\tilde{G} = \mathbf{C}^n$. Let π be the canonical projection $\tilde{G} \to G$. Then its kernel Γ is a discrete subgroup of the complex vector space $\mathbf{C}^n$, that is referred to the (associated) lattice of G. We can take a finite set of $\mathbf{R}$-linearly independent vectors $\{\gamma_i | i = 1, 2, \dots r\}$ together with $\{e_1, e_2, \dots, e_n\}$ form a base of $\tilde{G}$ over $\mathbf{R}$. Thus as a real Lie group G is isomorphic to the direct product group $\Pi_{i=1}^{n+r} S^1 \times \mathbf{R}^{n-r}$. So if $r = n$, G is the ordinary (compact) complex torus of complex dimension n. If $0 < r < n$, then G is a non-compact (H,C) group or toroidal group. If $r = 0$, then G is holomorphically equivalent to the algebraic torus $(\mathbf{C}^*)^n$ of complex dimension n , hence not (H,C). By Kazama Umeno [3] or Vogt [15] we know as cohomological conditions for an (H,C) group G, given in terms of the linear independence of the lattice generators over the rational number fields $\mathbf{Q}$.

4 The automorphism group of a toroidal group

Now we follow the notation in the preceding section. Let G be a an (H,C) group of complex dimension n with the lattice Γ of rank $m = n + r$. Here we investigate the structure of $Aut(G)$, where we consider G to be a complex space forgetting it is a complex Lie group. Thus an element $f \in Aut(G)$ has nothing to do with the group operation of G by definition. But an element $g \in G$ is naturally considered an element of $Aut(G)$ by translation. Thus, before stating a theorem, we prepare the following

Definition 4.1 (translation) *Translation* t of G is a holomorphic transformation of G that is given by the following mapping:

$$t(x) = x + a$$

for some element $a \in G$. To emphasize the dependence of t on an element a we denote t by t_a. The group of translations forms a subgroup of $Aut(G)$, called the translation group of G; $t : G \to Aut(G)$ defined as above is obviously an injective group homomorphism.

The natural question is whether it is surgective. Now we state

Theorem 4.1 (Abe [**?**] *Above homomorphism t is generically an isomorphism, or equivalently every holomorphic automorphism of G is given by some translation t_a for an element $a \in G$ up to finite symmetries.*

*Proof.*The readers are kindly referred to the forthcoming paper for the a generalization ([14]). The proof given here is essentially the same with the compact case, but we include it here for completeness. Anyway let us take an arbitrary automorphism f of G. Then taking the universal covering $\tilde{G}$, we get the lift $\tilde{f}$ of f. By definition $\tilde{f}$ is a holomorphic self-mapping of $\tilde{G}$. Since f is an automorphism f has a holomorphic inverse f^{-1}. Thus f^{-1} has a lift $\tilde{f}^{-1}$, which is a holomorphic self mapping of tdG. From the uniqueness of the covering mapping we have the identity $(\tilde{f})(\tilde{f}^{-1}) = id_{\tilde{G}}$, which shows the existence of the canonical homomorphism $\pi* : Aut(G) \to Aut(\tilde{G}); \pi * (f) = \tilde{f}$. Since $\tilde{f} : \tilde{G} \to \tilde{G}$ is invariant under the action of the lattice Γ, it satisfies the following equations:

$$\tilde{f}(z + \gamma) = \tilde{f}(z) + \gamma_z, \quad z \in \tilde{G},\, \gamma,\, \gamma_z \in \Gamma,$$

where γ_z is the additive automorphic factor with respect to the lattice Γ, which is not co-compact since we have only to consider a non-compact (H,C) group G. We can express the mapping $\tilde{f} : \mathbf{C}^n \to \mathbf{C}^n$ in terms of the coordinate functions $z_1, z_2, \ldots, z_n$ as follows:

$$\tilde{f}(z) = (\tilde{f}_1(z), \tilde{f}_2(z), \ldots, \tilde{f}_n(z)), \quad z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n.$$

We have the following commutative diagram:

$$\begin{array}{lcl} \tilde{G} = \mathbf{C}^n & \xrightarrow{\tilde{f}} & \tilde{G} = \mathbf{C}^n \\ \downarrow \pi & & \downarrow \pi \\ G = \mathbf{C}^n/\Gamma & \xrightarrow{f} & G = \mathbf{C}^n/\Gamma \end{array}$$

whose rows are isomorphism. The commutativity is shown above. Then each entire holomorphic function $\tilde{f}_j$ on $\mathbf{C}^n$ is not pushed down to the quotient holomorphic functionf_j on the lower row so that the following diagram be commutative:

$$\begin{array}{lcl} \tilde{G} = \mathbf{C}^n & \xrightarrow{\tilde{f}_j} & \mathbf{C} \\ \downarrow \pi & & \\ G = \mathbf{C}^n/\Gamma & \overset{f_j}{\nearrow} & \end{array}$$

But passing through to the respective partial differential of the function which can be well defined by using the additive automorphic factor which is reduced to the respective constant due to the discreteness of Γ. Then all of them are constant since the base space is (H,C). Thus we deduce f is a linear mapping, from which there follows f is a translation up to the signature of the respective partial derivative of each coordinate representation f_j. Q.E.D..

5 Final remarks on automorphisms of holomorphic principal bundles

Morimoto [9] obtained a result on automorphisms of holomorphic principal bundles:

Theorem 5.1 (Morimoto [9]) *Let $P := P(M, G)$ be a holomorphic principal bundle over a compact complex manifold M as a base space, with the fiber complex Lie group G. Then the automorphism group $Aut_F(P)$ is a complex Lie transformation group acting holomorphically on P. Furthermore we have the canonically induced complex Lie group homomorphism $\pi_* : Aut_F(P) \to Aut(M)$.*

He also proved the surjectivity of the above homomorphism under some additional conditions ;the one is the existence of a holomorphic connection of $P(M, G)$ and the other is the vanishing of the first Betti number of M.

Remark 5.1 In our setting the (H,C) group G is a holomorphically nontrivial holomorphic principal bundle $P(S, T)$ as was stated in the beginning theorem, where S is the maximal Stein subgroup which is normal in G since G itself is abelian and T is a compact complex torus obtained as a quotient group G/S. Then we have an exact sequence of complex Lie group as follows:

$$0 \to S \to G \to T \to 0$$

. Then we have the associated sequence

$$0 \to \ker \pi \to Aut_F(G) \to Aut(T) \to 0.$$

Obviously the second condition above is not satisfied but the first one is not examined yet. Anyway above theorem is a *partial* example of Morimoto's theorem and proposition in [9], except the connectivity of $Aut(T)$, since the second exact sequence is generically the same with the first one where $\ker(\pi)$ is equal to the typical fiber S, π denoting the natural surjection $G \to T = G/S$.

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