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# Generalization of Ceva's theorem on the triangle to the tetrahedron

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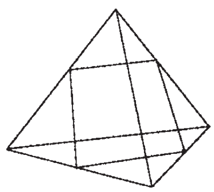
Abstract. We generalize Ceva's theorem on the triangle to the tetrahedron.

Let PQRS denote an arbitrary tetrahedron in Euclidean space $\mathbf{R}^n$ with $n \geq 3$, whose vertices are P, Q, R and S. Let $\mu$ denote the standard Euclidean measure (=volume) on $\mathbf{R}^3$. Let $m$ denote the standard Euclidean measure (=area) on $\mathbf{R}^2$. Let T be a point in the interior of PQRS. Let P' denote the point of the intersection of line PT and plane QRS. Let Q' denote the point of the intersection of line QT and plane PRS. Let R' denote the point of the intersection of line RT and plane PQS. Let S' denote the point of the intersection of line ST and plane PQR.



We obtain

**Theorem 1.**

1. $$\frac{m(\triangle \mathrm{P'SQ})m(\triangle \mathrm{P'QR})}{m(\triangle \mathrm{P'RS})^2} \cdot \frac{m(\triangle \mathrm{Q'PR})m(\triangle \mathrm{Q'RS})}{m(\triangle \mathrm{Q'SP})^2} \cdot \frac{m(\triangle \mathrm{R'QS})m(\triangle \mathrm{R'SP})}{m(\triangle \mathrm{R'PQ})^2} \cdot \frac{m(\triangle \mathrm{S'RP})m(\triangle \mathrm{S'PQ})}{m(\triangle \mathrm{S'QR})^2} = 1$$

2. $$\frac{m(\triangle \mathrm{P'SR})m(\triangle \mathrm{P'RQ})}{m(\triangle \mathrm{P'QS})^2} \cdot \frac{m(\triangle \mathrm{R'PQ})m(\triangle \mathrm{R'QS})}{m(\triangle \mathrm{R'SP})^2} \cdot \frac{m(\triangle \mathrm{Q'RS})m(\triangle \mathrm{Q'SP})}{m(\triangle \mathrm{Q'PR})^2} \cdot \frac{m(\triangle \mathrm{S'QP})m(\triangle \mathrm{S'PR})}{m(\triangle \mathrm{S'RQ})^2} = 1$$

3. $$\frac{m(\triangle \mathrm{P'RQ})m(\triangle \mathrm{P'QS})}{m(\triangle \mathrm{P'SR})^2} \cdot \frac{m(\triangle \mathrm{Q'PS})m(\triangle \mathrm{Q'SR})}{m(\triangle \mathrm{Q'RP})^2} \cdot \frac{m(\triangle \mathrm{S'QR})m(\triangle \mathrm{S'RP})}{m(\triangle \mathrm{S'PQ})^2} \cdot \frac{m(\triangle \mathrm{R'SP})m(\triangle \mathrm{R'PQ})}{m(\triangle \mathrm{R'QS})^2} = 1$$

*Proof of Theorem 1.* We give our proof of 1. Our proofs of 2 and 3 are the same as that of 1. We use

If $a:b:c = u:v:w = x:y:z$, then $a:b:c = u-x:v-y:w-z$.

Computing the volume of tetrahedrons and using this we have

$$m(\triangle \mathrm{P'RS}) : m(\triangle \mathrm{P'SQ}) : m(\triangle \mathrm{P'QR}) = \mu(\mathrm{PP'RS}) : \mu(\mathrm{PP'SQ}) : \mu(\mathrm{PP'QR})$$
$$= \mu(\mathrm{TP'RS}) : \mu(\mathrm{TP'SQ}) : \mu(\mathrm{TP'QR})$$
$$= \mu(\mathrm{TPRS}) : \mu(\mathrm{TPQS}) : \mu(\mathrm{TPQR}).$$

By the same method,
$m(\triangle \mathrm{Q'SP}) : m(\triangle \mathrm{Q'PR}) : m(\triangle \mathrm{Q'RS}) = \mu(\mathrm{TPQS}) : \mu(\mathrm{TPQR}) : \mu(\mathrm{TQRS})$,
$m(\triangle \mathrm{R'PQ}) : m(\triangle \mathrm{R'QS}) : m(\triangle \mathrm{R'SP}) = \mu(\mathrm{TPQR}) : \mu(\mathrm{TQRS}) : \mu(\mathrm{TPRS})$, and
$m(\triangle \mathrm{S'QR}) : m(\triangle \mathrm{S'RP}) : m(\triangle \mathrm{S'PQ}) = \mu(\mathrm{TQRS}) : \mu(\mathrm{TPRS}) : \mu(\mathrm{TPQS})$.
For simplicity, put $A = \mu(\mathrm{TQRS})$, $B = \mu(\mathrm{TPRS})$, $C = \mu(\mathrm{TPQS})$ and $D = \mu(\mathrm{TPQR})$.
Then the left side of 1 is equal to

$$\frac{CD}{B^2} \cdot \frac{DA}{C^2} \cdot \frac{AB}{D^2} \cdot \frac{BC}{A^2} = \frac{A^2B^2C^2D^2}{A^2B^2C^2D^2} = 1.$$

Hence 1 is proved.

By the same way, the left side of 2 is equal to $\frac{BD}{C^2} \cdot \frac{DA}{B^2} \cdot \frac{AC}{D^2} \cdot \frac{CB}{A^2} = 1$.

Hence 2 is proved.

By the same way, the left side of 3 is equal to $\frac{DC}{B^2} \cdot \frac{CA}{D^2} \cdot \frac{AB}{C^2} \cdot \frac{BD}{A^2} = 1$.

Hence 3 is proved.
Theorem 1 is proved.

Let I(PQ) denote the point of the intersection of line PQ and plane TRS. Let I(PR) denote the point of the intersection of line PR and plane TSQ. Let I(PS) denote the point of the intersection of line PS and plane TQR. Let I(RS) denote the point of the intersection of line RS and plane TPQ. Let I(QS) denote the point of the intersection of line QS and plane TPR. Let I(QR) denote the point of the intersection of line QR and plane TPS.

For arbitrary points U and V of $\mathbf{R}^n$ we denote by $|\mathrm{UV}|$ the length of vector $\overrightarrow{\mathrm{UV}}$.

We have the following corollary of Theorem 1.

**Corollary 1 of Theorem 1.**

1. $$\frac{|\mathrm{QI(QR)}|}{|\mathrm{I(QR)R}|} \cdot \frac{|\mathrm{RI(RS)}|}{|\mathrm{I(RS)S}|} \cdot \frac{|\mathrm{I(PS)S}|}{|\mathrm{PI(PS)}|} \cdot \frac{|\mathrm{I(PQ)P}|}{|\mathrm{QI(PQ)}|} = 1.$$

2. $$\frac{|\mathrm{RI(QR)}|}{|\mathrm{I(QR)Q}|} \cdot \frac{|\mathrm{QI(QS)}|}{|\mathrm{I(QS)S}|} \cdot \frac{|\mathrm{I(PS)S}|}{|\mathrm{PI(PS)}|} \cdot \frac{|\mathrm{I(PR)P}|}{|\mathrm{RI(PR)}|} = 1.$$

3. $$\frac{|\mathrm{QI(QS)}|}{|\mathrm{I(QS)S}|} \cdot \frac{|\mathrm{SI(RS)}|}{|\mathrm{I(RS)R}|} \cdot \frac{|\mathrm{I(PR)R}|}{|\mathrm{PI(PR)}|} \cdot \frac{|\mathrm{I(PQ)P}|}{|\mathrm{QI(PQ)}|} = 1.$$

*Proof of Corollary 1.*
From 1 of Theorem 1, we obtain

$$\frac{|QI(QR)|}{|I(QR)R|}\cdot\frac{|QI(QS)|}{|I(QS)S|}\cdot\frac{|RI(RS)|}{|I(RS)S|}\cdot\frac{|I(PR)R|}{|PI(PR)|}\cdot\frac{|I(PS)S|}{|PI(PS)|}\cdot\frac{|I(QS)S|}{|QI(QS)|}\cdot\frac{|I(PQ)P|}{|QI(PQ)|}\cdot\frac{|I(PR)P|}{|RI(PR)|}=1.$$

This implies 1 of Corollary 1. Changing R and Q into Q and R respectively in 1 of Corollary 1, we have 2 of Corollary 1. Changing R and S into S and R respectively in 1 of Corollary 1, we have 3 of Corollary 1. Corollary 1 is proved.

We also obtain

**Theorem 2.**

1. $$\frac{m(\triangle P'QR)m(\triangle P'RS)}{m(\triangle P'QS)^2}\cdot\frac{m(\triangle Q'PS)m(\triangle Q'RS)}{m(\triangle Q'PR)^2}\cdot\frac{m(\triangle R'PQ)m(\triangle R'PS)}{m(\triangle R'QS)^2}\cdot\frac{m(\triangle S'PQ)m(\triangle S'QR)}{m(\triangle S'PR)^2}=1$$

2. $$\frac{m(\triangle P'QR)m(\triangle P'QS)}{m(\triangle P'RS)^2}\cdot\frac{m(\triangle R'PS)m(\triangle R'QS)}{m(\triangle R'PQ)^2}\cdot\frac{m(\triangle Q'PR)m(\triangle Q'PS)}{m(\triangle Q'RS)^2}\cdot\frac{m(\triangle S'PR)m(\triangle S'QR)}{m(\triangle S'PQ)^2}=1$$

3. $$\frac{m(\triangle P'QS)m(\triangle P'RS)}{m(\triangle P'QR)^2}\cdot\frac{m(\triangle Q'PR)m(\triangle Q'RS)}{m(\triangle Q'PS)^2}\cdot\frac{m(\triangle S'PQ)m(\triangle S'PR)}{m(\triangle S'QR)^2}\cdot\frac{m(\triangle R'PQ)m(\triangle R'QS)}{m(\triangle R'PS)^2}=1$$

*Proof of Theorem 2.*
The left side of 1 of Theorem 2 is equal to $\frac{DB}{C^2}\cdot\frac{CA}{D^2}\cdot\frac{DB}{A^2}\cdot\frac{CA}{B^2}=1$.
The left side of 2 of Theorem 2 is equal to $\frac{DC}{B^2}\cdot\frac{BA}{D^2}\cdot\frac{DC}{A^2}\cdot\frac{BA}{C^2}=1$.
The left side of 3 of Theorem 2 is equal to $\frac{CB}{D^2}\cdot\frac{DA}{C^2}\cdot\frac{CB}{A^2}\cdot\frac{DA}{B^2}=1$.
Theorem 2 is proved.

APPENDIX. We write here classical Ceva's theorem.

**Ceva's Theorem.** Let $\triangle$PQR be an arbitrary triangle in Euclidean space $\mathbf{R}^n$ with $n\geq 2$. Let W denote a point in the interior of $\triangle$PQR. Let L denote the point of the intersection of lines PW and QR. Let M denote the point of the intersection of the lines QW and PR. Let N denote the point of the intersection of the lines RW and PQ. Then one has

$$\frac{|NQ|}{|PN|}\cdot\frac{|LR|}{|QL|}\cdot\frac{|MP|}{|RM|}=1.$$