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On some aspects of non-euclidean plane geometries

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Abstract

The purpose of the present short note is to investigate some aspects of non-euclidean plane geometry. As an advantage of plane geometry we could give some explicit illustration of the obtained results [7]. As is known by Buseman [1] there is an axiomatization of non-riemannian geometry in terms of geodesics,*i.e.*, G-space. Here we are concerned about metric functions on the real plane $\boldsymbol{R}^2$ with no additional axiom. There are obtained a few results and many of the important problems are still open and are given in the paper. [1]

1 Introduction

In any abstract set X we have the most simple example of metric function, *i.e.*, the discrete metric. We could consider here as the abstract set X the real plane $\boldsymbol{R}^2$ and then the non-empty set $\mathcal{M}(X)$ of metric functions on X. Since the real numbers set $\boldsymbol{R}$ has the linear order $\leq$, we have the induced (partial) order $\leq$ in $\mathcal{M}(X)$ as a subset of $\boldsymbol{R}^X$, where Y^X denotes the set of mappings of X to Y, henceforth. Further it has the additive structure and the cone structure induced from $\boldsymbol{R}^X$, since it is closed under the additive operation and the positive scalar multiplication. It has further the poset structure; for two metrics d_1, d_2, we define $d_1 \leq d_2$ iff $d_1(x) \leq d_2(x)$ for any x. This poset structure on $\mathcal{M}(X)$ is compatible with the additive structure, hence it is a partially ordered additive semigroup. If two metrics d_1, d_2 are proportional they are defined to be equivalent. Hence we obtain the quotient set $\mathcal{PM}(X)$ of $\mathcal{M}(X)$ by this equivalence relation. We call the equivalence class of a metric d the projective class and denote it $[d]$. Some times we call $[d]$ simply metric without an additional phrase "projective class of" if there is no fear of confusion. In the following sections we will investigate a specified sequence of metrics d_n defied as follows:

$$d_n(x,y) = \{\sum_{i=1}^{2} |x_i - y_i|^n\}^{1/n},\ n = 1, 2, 3, \ldots$$

for $x = (x_1, x_2), y = (y_1, y_2) \in X$. The classical euclidean (plane) geometry is referred to the case of $n = 2$, and is well known hence omitted. In the next section we will treat the case of $n = 1$. The other cases are not treated at all, but it is proved that they all satisfy the metric axioms, which fact could be found in the literature. We should close this section by eventually remarking that the sequence $\{d_n\}_{n=1}^{\infty}$ converges to the limit metric d_∞ defined as

$$d_\infty(x,y) := \sup\{|x_i - y_i|; i = 1, 2\},$$

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where $\mathcal{M}(X)$ is assumed to have the natural bounded supremum metric D defined as:

$$D(d_1, d_2) := \sup\{|\tilde{d}_1(x,y) - \tilde{d}_2(x,y)|; x, y \in X\}.$$

Here $\tilde{d}$ denotes the metric defined as $\tilde{d}(x,y) := \dfrac{d(x,y)}{1+d(x,y)}$. Then $\mathcal{M}(X)$ with this metric D is the metric space. In this respect the (projective class of) discrete metric is the extremal one.

Definition 1.1 (degree of metric in $\mathcal{PM}(X)$) A metric d in $\mathcal{PM}(X)$ is of degree n, denoted $n = \deg(d)$, iff d is equivalent to d_n or simply we say d is a *degree-n* metric.

Definition 1.2 (non-euclidean geometry) A metric d is *euclidean* if $\deg(d) = 2$, and the geometry of the metric space $(\boldsymbol{R}^2, d)$ is the euclidean (plane) geometry. Other geometry is hence non-euclidean.

2 Non-smooth Finsler geometry

In this section we should mention the relation between our non-euclidean geometry and the classical Finsler geometry ([2]). As compared to the riemannian case Finsler metric is defined on a smooth manifold M by giving a (smooth) so-called *norm function* on the tangent bundle TM of M. If this norm function is induced from some (smooth) inner product function on TM, then it defines a riemannian metric; on Finsler space we can define only an infinitesimal length of a (smooth) curve, and not the angle of two (tangent) vectors as an invariant quantity, as is the case in the normed vector space in general. We should here clarify the relation between (non-euclidean) metric functions and lengthes of tangent vectors especially for the case of degree one metric d_1 on the non-euclidean plane $X = \boldsymbol{R}^2$. Let us take a point $x \in X$ with the standard global coordinates (x_1, x_2). Then the tangent space $T_x = T_x(X)$ of X at x is generated by the basis $\{\partial_1, \partial_2\}$, where $\partial_i = \dfrac{\partial}{\partial_{x_i}}, i = 1, 2$. According to our method of measurement of the distance between two points $x, a \in X$, where $a = (a_1, a_2)$, we have $d_1(x, a) = |x_1 - a_1| + |x_2 - a_2|$. This implies the norm function, denoted n, is defined by

$$n(\textstyle\sum_{i=1}^{2} y_i \partial_i) = \sum_{i=1}^{2} |y_i|,$$

where $y = (y_1, y_2)$ is the standard (global) fiber coordinates on the tangent bundle space TX, which is trivial (globally). Exactly speaking the norm function n is a (continuous) function on the tangent bundle TX. Hence it may depend on the variable $x \in X$ in general, but not the case for our metric d_1 since the right hand quantity in the above defining equation doesn't contain the x-term as y_i's are globally defined. But it is easily observed to be non-smooth, even not $\mathcal{C}^1$ regular, hence not a finsler metric in the original sense. If we extend the notion of Finsler geometry to the non-smooth case, our non-euclidean plane might be treated in that (extended) category. But for the present moment we have few remarkable specific features shared by the classical (smooth) Finsler geometry. We could give, among others, the typical one; the distance between two points in Finsler space is attained by the length of a shortest geodesic, which does exist if the space is complete w.r.t. the metric. This is given by the solving the variational problem, which is not yet formulated in our space (X, d_1). By analogy to the riemannian case the line element ds of the metric function d_1 of $X = \boldsymbol{R}^2$ is expressed as

$$ds = |dx_1| + |dx_2|,$$

or equivalently we have

$$ds^2 = dx_1^2 + dx_2^2 + 2|dx_1 dx_2|.$$

The other example is that in Finsler geometry the angle of two vectors are not defined coherently, since the norm function gives only the length of curves; but a "Finsler" connection is introduced by E.Cartan [3] as a generalization of riemannian case. In the next section we observe the situations in our space (X, d_1), and will try to define geodesics or perpendicular normal.

3 geodesics and arc length in d_1 geometry

In this section we consider only the metric space X with an arbitrary metric d, without any additional condition. So X may be or may not be euclidean or riemannian. If it is necessary to restrict to the noneuclidean plane of degree 1, we will say so. First let us denote by I any connected subset of the real line $\boldsymbol{R}$, with the usual topology. Then first we give

Definition 3.1 (geodesic [1]) A continuous mapping $\gamma : I \to X$ is a *geodesic*, if for every point $x = \gamma(t_0)$ on the curve $\gamma(I)$ and for any sufficiently small number $s, t \in \boldsymbol{R}$ such that $t_0 + s, t_0 + s + t \in I$), we have

$$d(x, \gamma(t_0 + s + t)) = d(x, \gamma(t_0 + s)) + d(\gamma(t_0 + s), \gamma(t_0 + s + t)).$$

A geodesic $\gamma : I = [0, 1] \to X$ have end points p=$\gamma(0), q = \gamma(1)$. Then we call such one geodesic with end points or $wepgeodesic, whichmightbecalledclosedgeodesic.$
It is easily observed that every (half) line/segment parallel (in the euclidean sense) to x or y axis is a geodesic. Hence we have

Definition 3.2 (axial geodesic) A geodesic $\gamma : I \to X$ is *axial* if the image curve $\gamma(I)$ is parallel to either x or y -axis. We call them axial geodesics.

There arises the following natural
Question 1. For given two points how many wep geodesics are connecting them?

As for this question Buseman [1] axiomatized G-space so that it has always wep geodesics (*line segment* in his terminology) with arbitrary given end points. It is not known at the present moment to the author if his axioms are necessary for wep geodesics to exist. One of the aims of the present paper is to show an example of non euclidean plane geometry, and we have the following answer whose proof is easy, hence omitted.

Theorem 3.1 *Let (X, d_1) be the non-euclidean plane of degree 1. Then any two points have (uncountably-) infinitely many wep geodesics in general. The only exceptional cases are when two points lie on an axial geodesic.*

For two wep-geodesics $\gamma_i, i = 1, 2$ with an appropriate common end point,*i.e.*e.g, $\gamma_1(1) = \gamma_2(0)$ we have their multiplication $\gamma = \gamma_1 \cdot \gamma_2$ in the usual fashion. This binary operation is inductively defined, and hence we have the multiplication of finite family $\{\gamma_i | i = 1, 2, \dots, n\}$ of ordered wep-geodesics with common end points of the ordered pairs of them, *e.g.*, $\gamma_i(1) = \gamma_{i+1}(0)$. We can easily verify that the resulting curve satisfies the geodesic condition mentioned above.

Definition 3.3 We call the above family a finite *admissible family* of wep-geodesics. Then a multiplication of finite appropriate wep geodesic families are *coherently oriented* if a γ_i is upward/downward or oriented to the right/left, then the next one γ_{i+1} is so too, for any index $1 \leq i \leq n$ and vice versa.

Any axial wep-geodesic is a geodesic and further as an immediate corollary to the above definition, we have

Theorem 3.2 *The multiplication of a finite admissible family of wep geodesics is a geodesic.*

General axial geodesic (with end points) is by definition, a multiplication of finite admissible family of axial wep geodesics.

Remark 3.1 Since the geodesic condition given above is apparently local, we could not prove the global minimality as is the case in Riemannian geometry. So to get the minimal geodesic in the sense it attains the shortest distance between the two end points along the curve, we need some additional condition: we have

Theorem 3.3 *A (general) wep-geodesic is minimal iff it is coherently oriented.*

We will give here no proof of this theorem, since it requires only elementary arguments . As for the length of curves we have the following

Definition 3.4 (arc length) Let $C : I \to \boldsymbol{R}$ be a curve with end points p, q. Then we define the length $l(C)$ of C by

$$l(C) = \sup\{\sum_{i=1}^{n} d(x_i, x_{i+1}) | n \in \mathbf{N}\},$$

where the points $\{x_i\}_{i \in \mathbf{N}}$ lie consecutively (in the natural order) on the curve C.

We should remark that for every minimal wep geodesic C this length coincides with the distance between the end points [1, 7].

Definition 3.5 (perpendicular bisector) Given two points $a, b \in X$, *perpendicular bisector* of a, b, denoted $PB_{a,b}$, is the set defined as:

$$PB = PB_{a,b} := \{x \in X | d(x, a) = d(x, b)\}$$

Remark 3.2 This definition follows the counterpart in euclidean geometry and might be a sort of abuse of terminology; the set PB is not necessarily a line and the perpendicularity is the euclidean notion, not yet established in the non-euclidean case. By the way a notion of (general) angle is introduced in [1], but we don't know yet the perpendicularity is expressed in his terminology or not, even if our space to satisfy his axiom of G-space.

Question 2. ([1, 2, 3]) To what extent our geodesics or perpendicular bisector share similar feature with those in the euclidean/riemannian (plane) geometries? Or further compare with the case of G-space of dimension two ([1, 2, 3]).

We could get similar results in d_∞ case, which would be, however, given in the forthcoming paper ([7]).

4 Isometry groups and related problems in non-euclidean geometry

We can formulate some problems in the non-euclidean geometry([4, 7]). First we define an isometry as

Definition 4.1 (isometry) Let $(X, d_X), (Y, d_Y)$ be metric spaces. A mapping $g : X \to Y$ is *distance-decreasing* if for any points $p, q \in X$, we have

$$d_Y(g(p), g(q)) \leq d_X(p, q)$$

, and conversely it is *distance-increasing* if the converse inequality holds. It is an *isometry* if it is at once distance-increasing and distance-decreasing, or if we have the equality $d_Y(g(p), g(q)) = d_X(p, q)$ for any $p, q \in X$,

The set of all the self isometries of X forms a group, called the *isometry group* G of X, denoted $G = Iso(X)$, with the multiplication defined by the composition of two isometries. Then it would be natural to ask the following

Question What is $Iso(X)$? Is it a Lie transformation group (with the compact open topology) which acts smoothly on X ? If so, determine (the dimension of) the Lie algebra of G.

There are no answer in the most general setting of the metric space X. However we can give a result in our setting. Before giving an answer to this question we need the following

Definition 4.2 (parallel translation) *Parallel translation* t of X is a transformation of X that is given by the following mapping:

$$t(x) = x + a,$$

for some element $a \in X$. The group of (all) the parallel translation forms a subgroup $T(X)$ of G, called the parallel translation group of X.

We should apologize for using the terminology *parallel* without defining it in our category. This is used in the euclidean geometry and we followed the usage temporarily.

Theorem 4.1 *The subgroup $T(X)$ is a normal subgroup of G, isomorphic to the vector group $\boldsymbol{R}^2$ of dimension 2.*

Definition 4.3 (orthogonal group) By analogy of the euclidean geometry we define a (non-euclidean) *orthogonal transformation* of X to be an element $g \in Iso(X)$ that fixes the origin $O \in X$. Then we have a subgroup $O(X)$ of G, called the *orthogonal group* of X.

$$O(X) = \{g \in G | g(O) = O\}$$

Remark 4.1 We could give some features of $O(X)$; it contains a finite cyclic abelian group $SO(X)$ of order 4, generated by the orthogonal rotation ρ as a primitive element. This group is a subgroup of the orientation preserving isometry group of X, denoted $SI(X)$, in the usual sense of the smooth manifold X.

Now we can give an answer to the above question, which might be generalized to higher dimensional cases with some minor modifications.

Theorem 4.2 *For a non-euclidean plane (X, d) of degree one we have*

$$Iso(X) = T(X) \cdot O(X),$$

which induces the semi direct product of G.

We could give here an outline of a proof of the theorem, but the readers are asked to see the forthcoming paper for the details ([7]).

5 Ellipses and other elementary figures in non-euclidean geometry

We will give various figures of ellipses with some illustrations at the appendix in [7]. First we give a

Definition 5.1 (ellipse) An ellipse with foci $a, b \in X$ and radius r is the set $E_{a,b;r}$ defined by

$$E_{a,b;r} := \{x \in X | d(x, a) + d(x, b) = 2r\}.$$

As a special case of ellipse for $a = b$ we have the definition of 1-dimensional non-euclidean sphere ,*i.e.*, a circle with the center $a = b$ and radius r.

We could give also some elementary figures familiar to every one in euclidean geometry, among others, parabola or hyperbola ([7]).

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