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Orthocenters of n dimensional simplexes

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Introduction

Let n be any integer ≥ 2, and let $P_0P_1\cdots P_n$ denote an n dimensional simplex in the Euclidean space $\mathbf{R}^n$ whose vertices are $\{P_0, P_1, \cdots, P_n\}$. In the case of n=2, it is well known that any triangle (namely, 2 dimensional simplex) has a unique orthocenter. We treat any n dimensional simplex in this note.

For an n dimensional simplex $P_0P_1\cdots P_n$, put $\vec{p}_j = \overrightarrow{P_0P_j}$ for any integer j with $1 \leq j \leq n$. Let $(\cdot,\cdot)$ denote the standard inner product of $\mathbf{R}^n$. For each integer j with $0 \leq j \leq n$, let π_j denote the hyper-plane (namely, $n-1$ dimensional linear submanifold of $\mathbf{R}^n$) which contains all of $\{P_x \mid 0 \leq x \leq n, x \neq j\}$ and let ℓ_j denote the line containing P_j and orthogonal to π_j. Since $P_0P_1\cdots P_n$ is an n dimensional simplex, it is easy to see

$$(\text{the number of the elements of} \bigcap_{j=0}^{n} \ell_j) \in \{0, 1\}.$$

If $(\text{the number of the elements of} \bigcap_{j=0}^{n} \ell_j) = 1$, the set $\bigcap_{j=0}^{n} \ell_j$ consists of one point and we call it the orthocenter of $P_0P_1\cdots P_n$.

In this note we present

Theorem 1. *Notations being as above, we have the following equivalence.*

$$(\text{the number of the elements of} \bigcap_{j=0}^{n} \ell_j) = 1$$

if and only if

$(\vec{p}_i, \vec{p}_j) = (\vec{p}_1, \vec{p}_2)$ *for all the integers i and j with* $i \neq j$, $1 \leq i \leq n$ *and* $1 \leq j \leq n$.

We prove Theorem 1 using Linear Algebra. We give two corollaries.

1. Proof of Theorem 1

(i) Assume that $(\text{the number of the elements of} \bigcap_{j=0}^{n} \ell_j) = 1$. Write $\{H\} = \bigcap_{j=0}^{n} \ell_j$. Put $\vec{p} = \overrightarrow{P_0H}$. Since $\ell_i \perp \pi_i$ for all $1 \leq i \leq n$, $(\vec{p} - \vec{p}_i, \vec{p}_j) = 0$ if $i \neq j$, $1 \leq i \leq n$ and $1 \leq j \leq n$. Hence $(\vec{p},\vec{p}_j)=(\vec{p}_i,\vec{p}_j)$ if $i \neq j$, $1 \leq i \leq n$ and $1 \leq j \leq n$. Hence $(\vec{p},\vec{p}_i)=(\vec{p}_j,\vec{p}_i)$ if $i \neq j$, $1 \leq i \leq n$ *and* $1 \leq j \leq n$. Hence $(\vec{p},\vec{p}_i)=(\vec{p},\vec{p}_j)$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$. (Then note that

$(\vec{p}, \overline{\vec{p}_i - \vec{p}_j})=0$ for all $1 \le i \le n$ and $1 \le j \le n$.) Hence we have $(\vec{p}_u,\vec{p}_v)=(\vec{p},\vec{p}_u)=(\vec{p},\vec{p}_i)=(\vec{p}_i,\vec{p}_j)$ if $i \ne j$, $u \ne v$, $1 \le i \le n$, $1 \le j \le n$, $1 \le u \le n$ and $1 \le v \le n$. Hence we have $(\vec{p}_i, \vec{p}_j) = (\vec{p}_1, \vec{p}_2)$ for all the integers *i* and *j* with $i \ne j$, $1 \le i \le n$ and $1 \le j \le n$.

(ii) Now assume $(\vec{p}_i, \vec{p}_j) = (\vec{p}_1, \vec{p}_2)$ for all the integers *i* and *j* with $i \ne j$, $1 \le i \le n$ and $1 \le j \le n$. Write $(\vec{p}_1, \vec{p}_2) = a$ for simplicity. We shall prove that

(*) there exists an *n*-dimensional vector $\vec{q}$ such that $(\vec{q} - \vec{p}_i, \vec{p}_j) = 0$ for all the integers *i* and *j* with $i \ne j$, $1 \le i \le n$ and $1 \le j \le n$.

If we assume (*), letting $\mathrm{Q} \in \mathbf{R}^n$ such that $\overrightarrow{\mathrm{P_0 Q}} = \vec{q}$, then we have $\overrightarrow{\mathrm{P}_i\mathrm{Q}} \perp \pi_i$ for all $1 \le i \le n$; and $(\vec{q}, \vec{p}_i - \vec{p}_j)=0$ for all $1 \le i \le n$ and $1 \le j \le n$. Then we have also $\vec{q} \perp \pi_0$, and Q becomes the orthocenter of $\mathrm{P_0 P_1 \cdots P}_n$.

We prove (*). We can write $\vec{q} = \sum_{j=1}^{n} \lambda_j \vec{p}_j$ with real numbers $\{\lambda_j\}_{j=1}^{n}$. (Note that $\mathrm{P_0 P_1 \cdots P}_n$ is an *n* dimensional simplex and that $\{\vec{p}_j\}_{j=1}^{n}$ are an **R**-basis of $\mathbf{R}^n$.) We shall show that the equations $(\sum_{j=1}^{n} \lambda_j \vec{p}_j, \vec{p}_u) = a$ for all $u \in \{y \mid 1 \le y \le n\}$, have real solutions $\{\lambda_j\}_{j=1}^{n}$. Since $\{\vec{p}_j\}_{j=1}^{n}$ are linearly independent over **R**, one has $\det((\vec{p}_i, \vec{p}_j))_{\substack{1 \le i \le n \\ 1 \le j \le n}} \ne 0$.

(Remark. Assume that $(\sum_{j=1}^{n} \mu_j \vec{p}_j, \vec{p}_u) = 0$ for all $u \in \{y \mid 1 \le y \le n\}$. Since $\{\vec{p}_j\}_{j=1}^{n}$ are an **R**-basis of $\mathbf{R}^n$, we have $\sum_{j=1}^{n} \mu_j \vec{p}_j = \vec{0}$ and $\mu_j = 0$ for all $j \in \{y \mid 1 \le y \le n\}$.) Hence the matrix $((\vec{p}_i, \vec{p}_j))_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ is invertible, and the the equations $(\sum_{j=1}^{n} \lambda_j \vec{p}_j, \vec{p}_u) = a$ for all $u \in \{y \mid 1 \le y \le n\}$ have unique real solutions $\{\lambda_j\}_{j=1}^{n}$. (*) is proven.

Theorem 1 is proven.

2. Corollaries of Theorem 1

We give two corollaries of Theorem 1.

Definition. Let $\mathrm{P_0 P_1 \cdots P}_n$ and $\mathrm{Q_0 Q_1 \cdots Q}_m$ be simplexes with $m \le n$. We call $\mathrm{Q_0 Q_1 \cdots Q}_m$ a subsimplex of $\mathrm{P_0 P_1 \cdots P}_n$ if and only if $\{\mathrm{Q}_x \mid 0 \le x \le m\} \subset \{\mathrm{P}_x \mid 0 \le x \le n\}$.

Corollary 1. *Let* $\mathrm{P_0 P_1 \cdots P}_n$ *be any n dimensional simplex with an orthocenter, and let m be any integer with* $2 \le m \le n$. *Any m dimensional subsimplex of* $\mathrm{P_0 P_1 \cdots P}_n$ *has an orthocenter.*

Corollary 2. *Let* $n \ge 3$. *Let m be any integer with* $3 \le m \le n$. *Fix m once for all. If each m dimensional subsimplex of an n dimensional simplex* $\mathrm{P_0 P_1 \cdots P}_n$ *has an orthocenter, then* $\mathrm{P_0 P_1 \cdots P}_n$ *has an orthocenter.*

We obtain these corollaries directly from Theorem 1 without difficulty.