

On a closed complexification of the G -manifold

by

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Abstract

The purpose of the present short note is to investigate a categorical correspondence between the category of compact real G -manifolds and that of (compact) complex H -manifolds, where G is the (infinite dimensional) real Lie group of automorphisms of a compact smooth manifold M and H is the complex Lie group of automorphisms of a closed complexification \hat{N} of M . Briefly speaking we are considering that the correspondence is not bijective in general.¹

1 Introduction

Let M be a compact smooth manifold of dimension m , G be the group of smooth automorphisms of M . Then without loss of generality we can assume M to be equipped with the unique real analytic structure compatible with the original smooth structure, and further G to be the group of real analytic automorphisms of M . In general G is an infinite dimensional real Lie group with the usual (compact open) topology. If M is embedded in some complex manifold \hat{N} of complex dimension m , \hat{N} contains a tubular neighbourhood $N = N(M)$ of M (in the sense of Grauert [4]) as an open submanifold of \hat{N} . Hereafter we assume \hat{N} to be compact, unless otherwise specified. The group H of complex automorphisms of \hat{N} has a (real closed) subgroup H_M , whose elements leave the submanifold M invariant. Though in general H_M doesn't carry a complex structure compatible with that of \hat{N} , it can be assumed to be a subgroup of G . We are concerned with the case when H_M is sufficiently big in the sense H_M acts transitively on M , which is the case if G acts transitively on M . Henceforth we investigate the category of G -homogeneous space M in the sense that a finite dimensional Lie (sub)group of G acts transitively on M . Let us take the (infinite dimensional) real Lie algebra \mathcal{G} of smooth vector fields on M . Since M is compact, every element of \mathcal{G} is complete, which generates the (connected component of the) group of smooth (global) transformations of M . We have a finite dimensional Lie subalgebra \mathcal{L} of \mathcal{G} which generates a finite dimensional (real) Lie group L of smooth automorphisms of M acting transitively on M . In the next section we study the 1 dimensional case as a prototype of higher dimensional cases. Before proceeding to the next section we need now to give

Definition 1.1 (*closed complexification*) Let M be a (connected) m -dimensional real analytic manifold, \hat{N} be a closed (connected) m -dimensional complex manifold. Then the pair (j, \hat{N}) of an (analytic) embedding $j : M \rightarrow \hat{N}$ and \hat{N} is a *closed complexification* of

¹Key Words and Phrases: complexification , G -manifold , automorphism group

M and j is the *associated closed embedding* of M . Conversely M is said to be a generic manifold of \hat{N} or simply a real form of \hat{N} ².

Categorically speaking we may say simply the morphism j itself is the *closed complexification* of the object M , where M is to belong to the category of real analytic manifolds.

Denoting the group of (biholomorphic) automorphisms of \hat{N} which leave M invariant by G_M , we have the associated group homomorphism $j^* : G_M \rightarrow G$ that is nothing but the restriction mapping. As is given in [4, 16] we can equip a certain tubular nbd $N = N(M)$ of M with an appropriate complex structure J , where $N(M)$ is identified with an open nbd TM of the zero section of the tangent bundle TM of M . This can be done ,e.g.by taking a riemannian metric g on M and then choosing a complex structure on the tangent bundle $T(TM)$ of TM with a hermitian metric h "induced" from the original metric g ([2, 4, 5, 8, 10, 15, 16]³. I will mention the matter in the appendix.).

Now we give a

Problem ([4, 8, 13, 15, 16, 17]) *For an m dimensional compact real analytic G -homogeneous manifold M , does there exist (at least) a closed H -homogeneous complexification (j, \hat{N}) ?*

Though in the full generality the above problem remains still open, we can and in fact do solve the problem in some restricted (*i.e.*, low dimensional) cases.

2 1 dimensional case

We will closely examine and prove the Theorem for one dimensional case. There exists only one (connected) object in one dimensional case in the category involved; the 1 dimensional sphere with the (unique) real analytic structure, *i.e.*, unit circle S^1 . The automorphism group $G = \text{Aut}(S^1)$ is infinite dimensional and its Lie algebra \mathcal{G} of infinitesimal automorphisms is given as follows:

$$\mathcal{G} = \{X = \sum f(\theta)\partial/\partial\theta | f \in \mathcal{C}^\omega(\mathbf{C})\}$$

where θ denotes the (local) coordinate of S^1 . Let us take any (real analytic) imbedding $j : S^1 \rightarrow \hat{N}$ of S^1 to a (compact) complex analytic curve \hat{N} , where N is a tubular neighbourhood of S^1 and \hat{N} is the *closed complexification* of S^1 . In the 1 dimensional case, *i.e.*, $M = S^1$, we can consider $TM = S^1 \times \mathbf{R}$ and hence $N = N(M)$ is identified topologically with an annulus in a complex line \mathbf{C} . We are involved only with the homogeneous case, hence the doubly connected region N may be identified with some of annuli in \mathbf{C} of a certain modulus (of open riemann surface in the classical moduli theory). So topologically there exist only two types of closed complexification N of M ; one is the projective line and the other is an elliptic curve, complex 1-dimensional torus. In the former case the group H is the projective transformation group $PGL(2, \mathbf{C})$ of ¹. For the notational simplicity we will represent an element of H by a representative element of the corresponding group $SL(2, \mathbf{C})$, if there is no fear of confusion. Hence the subgroup H_M which leaves invariant

²In somewhat different context it is said to be the center of $N = N(M)$ in [8]. Here we prefer this terminology to be compatible with that in the CR geometry.

³Making use of this opportunity I should mention that some of the remarks given in [16] about adapted complex structure are not precisely stated since there are confusions between the works of [2, 4, 5, 8, 10]

the submanifold M , the equator S^1 , in $\hat{N} = \mathbf{C}P^1$ is described as follows:

Let an element $z \in \mathbf{C}P^1$ be denoted by $z = [z_0, z_1]$ in homogeneous coordinates z_0, z_1 . Then we have $S^1 = \{z \in \mathbf{C}P^1 \mid |z_0| = |z_1|\}$. And furthermore we have

$$\begin{aligned} H_M &= \{g \in H \mid g(M) = M\} \\ &= \left\{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; gz = g[z_0, z_1] \in S^1\right\} \\ &= \left\{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; |az_0 + bz_1| = |cz_0 + dz_1|, \forall z \in S^1\right\} \\ &= \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; |a| = |d| = 1, ad = 1\right\} \end{aligned}$$

Eventually this implies that H_M is isomorphic to the unit circle S^1 in the complex line \mathbf{C} , as a multiplicative group. We have to show how to reconstruct the closed complexification (j, \hat{N}) with the group action H from the given data of the original (M, G) . On the tangent bundle TM of M we have the induced flat metric if we assume the flat metric on $M = S^1$. We have the Lie algebra \mathcal{G} of infinitesimal automorphism of M . Then we can extract the one dimensional subalgebra \mathcal{L} , which is described as the rotation along the circle by using the standard coordinate θ on M being identified with the standard unit sphere S^1 . Conversely the closed complexification $\mathbf{C}P^1$ and the projective transformation group $PGL(2, \mathbf{C})$ of $\mathbf{C}P^1$ are reconstructed as follows: take a tubular neighbourhood $N = N(M)$ of $M = S^1$ and choose a hermitian metric h on N so that h is an extension of a flat metric on S^1 and further each fiber is totally geodesic and the riemannian connection ∇ of h gives rise to the horizontal curve at each point on some (and hence on every) fiber, that is analytically diffeomorphic to the base manifold S^1 . We can choose the metric h so that the diameter of the horizontal circle decreases to zero as the corresponding point on the fiber goes apart from the zero section to ∞ . Finally we get a compactification \hat{N} of TM by adding the ∞ point at the both ends of each fiber (and hence all fibers) identified. It is not trivial to construct a complex analytic diffeomorphism $\phi : \hat{N} \rightarrow \mathbf{C}P^1$ and hence the details will be given elsewhere ([19]). Then there remains the problem to reconstruct the automorphism group H which should eventually be isomorphic to $PGL(2, \mathbf{C})$. For this purpose first we get the (complex) Lie algebra \mathcal{H} of all the holomorphic vector fields on \hat{N} in the sense of [14]. Then second we could have a Lie algebra isomorphism $\phi^* : \mathcal{H} \rightarrow \mathcal{PGL}(\infty, \mathbf{C})$ induced from ϕ equivariantly, we were done from the global topological reasoning.

In the latter case the connected component of the group H is an abelian variety of dimension one, complex torus \hat{N} itself. In comparison with the former case, closed complexifications have moduli of complex dimension one: the upper half plane in \mathbf{C} . This ambiguity occurs when we take a closed complexification \hat{N} from a tubular neighbourhood $N(M)$ of M . The precise statement of the procedure is as follows: We take the double of the tubular neighbourhood and patching together both of them according to the appropriate marking there. Hence this gives rise to the coordinate transformation on the covering shared by both boundary components, which are homeomorphic to the standard S^1 . This could be understood if we take various admissible riemannian metrics on M , which satisfy

one of the (sufficient) conditions for the existence of a compatible complexification of M ([2, 5, 4, 8, 10]).

3 Higher dimensional cases

According to the classical work by Tits, there are only five types of inequivalent compact homogeneous complex surfaces ([20]); it is either \mathbf{CP}^2 , $\mathbf{P}^1 \times \mathbf{P}^1$, a torus \mathbf{T} , a homogeneous Hopf surface \mathbf{H} or the product of an elliptic curve with \mathbf{P}^1 , all but for the former two cases having continuous moduli. Since all of elliptic curves and the rational one have a generic homogeneous real manifold M , we were done for the decomposable cases. The rests, *i.e.*, the irreducible cases must be treated separately. See the forthcoming paper [19] for the details.

4 Appendix

To the author's knowledge there are very few papers published on the mutual relations of various complexifications of a real (analytic/differentiable) manifold M , even in the compact case. For example it is not clear whether the Grauert tube coincides with the Lempert-Szöke complexification or that of Sasaki (there are no references on their bibliographies, even though one is considering the tangent bundle and the other the normal bundle). In some cases they should coincide, but in other cases they might differ. The problem of distinguishing these objects is to be the central theme of our forthcoming paper [19]

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