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Generalization of certain inequality for the triangle

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Introduction

Let S be any point in the interior of the triangle ABC. It is well known that

$$\frac{1}{2}(\mathrm{AB}+\mathrm{BC}+\mathrm{CA}) < \mathrm{SA}+\mathrm{SB}+\mathrm{SC} < \mathrm{AB}+\mathrm{BC}+\mathrm{CA}.$$

I propound

Problem 1. Generalize this to the n-simplex for all the integers $n \geq 2$.

Problem 1 seems proper as an exercise for students.

For a finite set X, write

#X=the number of all the elements of X and

$\mathcal{P}(X)$ *=the power set of X* .

My answer to this problem is the following Theorem 1.

Let n be any integer ≥ 2. Let $\mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_n$ denote any n-simplex in the Euclidean space $\mathbf{R}^n$ whose vertices are $\{\mathrm{P}_0,\mathrm{P}_1,\cdots,\mathrm{P}_n\}$. Let S be any point in the interior of $\mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_n$. Write

$$U = \{u \in \mathcal{P}(\{\mathrm{P}_0,\mathrm{P}_1,\cdots,\mathrm{P}_n\}) \mid \#u = n\} \quad \text{and}$$
$$V = \{v \in \mathcal{P}(\{\mathrm{P}_0,\mathrm{P}_1,\cdots,\mathrm{P}_n\}) \mid \#v = n-1\}.$$

Then we have $\#U = n+1$ and $\#V = \dfrac{n(n+1)}{2}$. Let μ be the usual measure on the $(n-1)$ dimensional Euclidean space. I obtain :

Theorem 1. *Notations being as above one has*

$$\frac{1}{2}\left(\sum_{u\in U}\mu((n-1)\text{-}\mathit{simplex\ whose\ vertices\ are\ } u)\right)$$
$$< \sum_{v\in V}\mu((n-1)\text{-}\mathit{simplex\ whose\ vertices\ are\ } v\cup\{\mathrm{S}\})$$
$$< \frac{n}{2}\left(\sum_{u\in U}\mu((n-1)\text{-}\mathit{simplex\ whose\ vertices\ are\ } u)\right).$$

Remark. In Theorem 1, note that

$$\sum_{u\in U}\mu((n-1)\text{-}\textit{simplex whose vertices are } u) = \mu(\textit{the boundary of } \mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_n).$$

In Section 1 I give my proof of Theorem 1.

In APPENDIX I correct misprints in Line 2 on Page 14 of my paper "p-Adic Properties of Sequences of Averages" in Science Reports of the Faculty of Education Gifu University (Natural Science), Vol. 26, No. 1, (2001), pp. 13-22.

1. Proof of Theorem 1

Let n be an integer ≥ 2. Letting t be a positive integer and $\{\mathrm{A}_j\}_{j=1}^{t}$ be points in $\mathbf{R}^n$ we write $<\{\mathrm{A}_j\}_{j=1}^{t}>$ for the minimum linear submanifold of $\mathbf{R}^n$ that contains $\{\mathrm{A}_j\}_{j=1}^{t}$. Namely

$<\{\mathrm{A}_j\}_{j=1}^{t}> = \{\mathrm{P}\in\mathbf{R}^n | \overrightarrow{\mathrm{A}_1\mathrm{P}} \in \textit{the } \mathbf{R}\textit{-linear subspace spanned by } \{\overrightarrow{\mathrm{A}_1\mathrm{A}_j}\}_{j=1}^{t}\}$.

We need

Lemma 1. *Let* $\mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_n$ *be any n-simplex in the Euclidean space* $\mathbf{R}^n$ *whose vertices are* $\{\mathrm{P}_0,\mathrm{P}_1,\cdots,\mathrm{P}_n\}$. *Then*

$$\mu(\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n) < \sum_{j=1}^{n}\mu(\textit{the } (n-1)\textit{-simplex whose vertices are } \{\mathrm{P}_0\}\cup\{\mathrm{P}_s \,|\, 1\leq s\leq n \text{ and } s\neq j\}).$$

Proof of Lemma 1. Let H denote the foot of the perpendicular from P_0 to $<\{\mathrm{P}_j\}_{j=1}^{n}>$. For each integer $j\in[1,n]$, let Q_j denote the foot of the perpendicular from H to $<\{\mathrm{P}_s \,|\, 1\leq s\leq n,\ s\neq j\}>$. We have $\mathrm{P}_0\mathrm{Q}_j^2 = \mathrm{Q}_j\mathrm{H}^2 + \mathrm{P}_0\mathrm{H}^2$, and $\mathrm{P}_0\mathrm{Q}_j > \mathrm{Q}_j\mathrm{H}$ since $\mathrm{P}_0\mathrm{H} > 0$. Hence we have

$\mu(\mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_{j-1}\mathrm{P}_{j+1}\cdots\mathrm{P}_n) > \mu(\mathrm{H}\mathrm{P}_1\cdots\mathrm{P}_{j-1}\mathrm{P}_{j+1}\cdots\mathrm{P}_n)$ for each integer $j\in[2,n]$, and

$\mu(\mathrm{P}_0\mathrm{P}_2\mathrm{P}_3\cdots\mathrm{P}_n) > \mu(\mathrm{H}\mathrm{P}_2\mathrm{P}_3\cdots\mathrm{P}_n)$.

The support of $\mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_n$ is a closed convex subset.

$\bigcup_{j=2}^{n}\textit{the support of } \mathrm{H}\mathrm{P}_1\cdots\mathrm{P}_{j-1}\mathrm{P}_{j+1}\cdots\mathrm{P}_n \bigcup \textit{the support of } \mathrm{H}\mathrm{P}_2\mathrm{P}_3\cdots\mathrm{P}_n$ is also closed and convex, and contains $\{\mathrm{P}_1,\mathrm{P}_2,\cdots,\mathrm{P}_n\}$. Hence

the support of $\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n$

$$\subset \bigcup_{j=2}^{n}\textit{the support of } \mathrm{H}\mathrm{P}_1\cdots\mathrm{P}_{j-1}\mathrm{P}_{j+1}\cdots\mathrm{P}_n \bigcup \textit{the support of } \mathrm{H}\mathrm{P}_2\mathrm{P}_3\cdots\mathrm{P}_n .$$

Hence $\mu(\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n) \leq \mu(\mathrm{H}\mathrm{P}_2\mathrm{P}_3\cdots\mathrm{P}_n) + \sum_{j=2}^{n}\mu(\mathrm{H}\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_{j-1}\mathrm{P}_{j+1}\cdots\mathrm{P}_n)$. Hence

$$\mu(\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_n) \leq \mu(\mathrm{H}\mathrm{P}_2\mathrm{P}_3\cdots\mathrm{P}_n) + \sum_{j=2}^{n}\mu(\mathrm{H}\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_{j-1}\mathrm{P}_{j+1}\cdots\mathrm{P}_n) < \sum_{j=1}^{n}\mu(\mathrm{P}_0\mathrm{P}_1\mathrm{P}_2\cdots\mathrm{P}_{j-1}\mathrm{P}_{j+1}\cdots\mathrm{P}_n),$$

QED.

Definition 1. Let n be any integer >1. Let $P_0P_1\cdots P_n$ be any n-simplex in $\mathbf{R}^n$ whose vertices are $\{P_0,P_1,\cdots,P_n\}$. For each integer $j\in[1,n]$, write Λ_j for the set $\{(n-1)$-simplex in $\mathbf{R}^n$ | Its vertices contains P_j and are contained in $\{P_0,P_1,\cdots,P_n\}\}$. We define $\mu(P_j;P_0P_1\cdots P_n)$ to be $\sum_{\lambda\in\Lambda_j}\mu(\lambda)$.

Let $P_0P_1\cdots P_n$ be any n-simplex in the Euclidean space $\mathbf{R}^n$ whose vertices are $\{P_0,P_1,\cdots,P_n\}$. Let S be a point in the interior of $P_0P_1\cdots P_n$. Put $S_0=S$. Inductively put

$S_1=$ *the unique point in* $<\{P_1,S\}>\cap<\{P_s|0\le s\le n, s\ne 1\}>$;

$S_2=$ *the unique point in* $<\{P_2,S_1\}>\cap<\{P_s|0\le s\le n, s\ne 2\}>$;

$S_j=$ *the unique point in* $<\{P_j,S_{j-1}\}>\cap<\{P_s|0\le s\le n, s\ne j\}>$ for $2\le j\le n$.

Then we have $\{S_0,S_1,\cdots,S_n\}$. We have also

$S_j=$ *the unique point in* $<\{P_j,S_{j-1}\}>\cap<\{P_s|0\le s\le n, s\notin\{x\in\mathbf{Z}|1\le x\le j\}\}>$

for any integer $j\in[1,n]$.

From these we have

$$\mu(S_j;S_jP_1P_2\cdots P_n)$$
$$=\mu(S_j;S_jS_{j-1}P_1P_2\cdots P_{j-1}P_{j+1}\cdots P_n)+(\mu(S_{j-1};S_{j-1}P_1P_2\cdots P_n)-\mu(S_{j-1}P_1P_2\cdots P_{j-1}P_{j+1}\cdots P_n))$$

for any integer j>1, and

$$\mu(S_1;S_1P_1P_2\cdots P_n)$$
$$=\mu(S_1;S_1S_0P_2P_3\cdots P_n)+(\mu(S_0;S_0P_1P_2\cdots P_n)-\mu(S_0P_2P_3\cdots P_n))$$

for j=1.

On the other hand from Lemma 1 we have

$\mu(S_j;S_jS_{j-1}P_1P_2\cdots P_{j-1}P_{j+1}\cdots P_n)-\mu(S_{j-1}P_1P_2\cdots P_{j-1}P_{j+1}\cdots P_n)>0$ for any integer j>1, and

$\mu(S_1;S_1S_0P_2P_3\cdots P_n)-\mu(S_0P_2P_3\cdots P_n)>0$ for j=1.

From these we have $\mu(S_j;S_jP_1P_2\cdots P_n)>\mu(S_{j-1};S_{j-1}P_1P_2\cdots P_n)$ for any integer j with $n\ge j\ge 1$. Recall $S_0=S$ and $S_n=P_0$. Hence we have proved

Proposition 1. *Let* $P_0P_1\cdots P_n$ *be any n-simplex in the Euclidean space* $\mathbf{R}^n$ *whose vertices are* $\{P_0,P_1,\cdots,P_n\}$. *Let* S *be a point in the interior of* $P_0P_1\cdots P_n$. *Then one has*

$$\mu(S;SP_1P_2\cdots P_n)<\mu(P_0;P_0P_1P_2\cdots P_n).$$

Recall

$U=\{u\in\mathcal{P}(\{P_0,P_1,\cdots,P_n\})\,|\,\#u=n\}$ and

$V=\{v\in\mathcal{P}(\{P_0,P_1,\cdots,P_n\})\,|\,\#v=n-1\}$.

Note $\#U=n+1$. We have

$$\sum_{u\in U}\mu(\text{S; the } n\text{-simplex whose vertices are } \{S\}\cup u)$$
$$=2\Big(\sum_{v\in V}\mu((n-1)\text{-}\textit{simplex whose vertices are } v\cup\{S\})\Big)$$

easily. By Lemma 1,

$\mu(\mathrm{S};\textit{the } n\textit{-simplex whose vertices are } \{\mathrm{S}\}\cup u) > \mu((n-1)\textit{-simplex whose vertices are } u)$

for any $u \in U$. Hence we have

$$2\Big(\sum_{v\in V}\mu((n-1)\textit{-simplex whose vertices are } v\cup\{\mathrm{S}\})\Big)$$

$$> \sum_{u\in U}\mu((n-1)\textit{-simplex whose vertices are } u).$$

On the other hand from Proposition 1 we have

$$\sum_{u\in U}\mu(\mathrm{S};\textit{the simplex whose vertices are } \{\mathrm{S}\}\cup u) < \sum_{j=0}^{n}\mu(\mathrm{P}_j;\mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_n).$$

Recall that *this left side* $= 2\Big(\sum_{v\in V}\mu((n-1)\textit{-simplex whose vertices are } v\cup\{\mathrm{S}\})\Big)$.

We have easily $\sum_{j=0}^{n}\mu(\mathrm{P}_j;\mathrm{P}_0\mathrm{P}_1\cdots\mathrm{P}_n) = n\left(\sum_{u\in U}\mu((n-1)\textit{-simplex whose vertices are } u)\right)$.

Hence we have

$$2\Big(\sum_{v\in V}\mu((n-1)\textit{-simplex whose vertices are } v\cup\{\mathrm{S}\})\Big)$$

$$< n\left(\sum_{u\in U}\mu((n-1)\textit{-simplex whose vertices are } u)\right).$$

Theorem 1 is proved.

APPENDIX

Correction of the misprints in Line 2 on Page 14 of Kazuyuki Hatada, "*p*-Adic Properties of Sequences of Averages" in Science Reports of the Faculty of Education Gifu University (Natural Science), Vol. 26, No. 1, (2001), pp. 13-22.

Read " $d_n = \lim_{h\to\infty}\frac{1}{fq^h}\sum_{\substack{a=1\\(p,a)=1}}^{fq^h}\chi(a)a^n$ " for " $d_n = \lim_{n\to\infty}\frac{1}{fq^n}\sum_{\substack{a=1\\(p,a)=1}}^{fq^n}\chi(a)a^n$ ".

References

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