岐阜大学教育学部研究報告（自然科学）第28巻第2号，79-88，2004年3月
Sci. Rep. Fac. Educ. Gifu Univ. (Nat. Sci.) 28 (2), 79-88, Mar. 2004

On the CR real form of complex Lie group action [1]

(Received on January 10th, revised on February 12th, 2004)

by
Shigeru Takeuchi [2]

Department of Mathematics, Faculty of Education, Gifu University
Yanagido 1-1, Gifu, Japan
email: takeuchi@cc.gifu-u.ac.jp

Contents

Abstract

The purpose of this paper is first to give a notion of the complexificationof the CR Lie group action on CR manifold and then to prove the existence and uniqueness theorem under some mild assumptions and second to give a notion of the CR real form of the complex Lie group action on complex manifold . The problem to prove the existence and uniqueness theorem for the CR real form is rather difficult except for some special cases and remains open in this paper. To prove the former theorem we needs a method of orbit decompositions under the compatibility condition, which is not explicitly stated here, where we refer the possible relations with the method of adapted complex structure on some quotient vector bundle of the tangent bundle, of real analytic complete CR kähler manifolds: We refer the readers to [2, 3, 6, 7, 8, 12] for more details of the (finite dimensional) CR category as basic ingredients.

---

[1] Key words: CR Lie group, CR real form, categorical complexification

[2] Partially supported by Grant in Aid for Scientific Research No.12640167 from Ministry of Education and Science, Japan.

1 CR category and complex category

1.1 Linear CR category

First let us recall, for the readers' convenience, a *CR vector space* is a triplet (V, R, J) of a real vector space V, a subspace R of V and a complex structure J on R. An $\mathbf{R}$-linear mapping $f : V_1 \to V_2$ of a CR space $V_1 = (V_1, R_1, J_1)$ to another one $V_2 = (V_2, R_2, J_2)$ is a *(CR) morphism* if f preserves the CR structures R_i, *i.e.*, $f(R_1) \subset R_2$ and f commutes the complex structures J_i, *i.e.*, $fJ_1 = J_2 f$. Let U, V and W be CR spaces. An $\mathbf{R}$-bilinear mapping $f : U \times V \to W$ is a *CR bilinear morphism* if f induces the CR morphisms to W for each fixed element of the respective factors U, V.

1.2 Complex representation of CR vector space

Letting $i = \sqrt{-1}$, we have the following complex representation for a CR vector space $V = (V, R, J)$; the usual complexification $V^c := V \otimes_{\mathbf{C}}$ of V, considered as a real vector space, has the direct sum splitting

$$V^c := V \otimes_{\mathbf{C}} = (R \oplus S)^c = H \oplus A \oplus S^c,$$

where S is a complementary direct summand of R in V with $H = H_V = \{x - iJx \in R^c | x \in R\}$ and $A = A_V = \overline{H} = \{x + iJx | x \in R^c\}$ being the $\pm i$ eigenspace of J respectively. We call $(A)H$ the (anti) holomorphic subspace of R^c, respectively. Conversely if we are given a direct sum splitting $H \oplus \overline{H}$ of a complex subspace of a complexified vector space V^c of V, then we have a unique real endomorphism $J : R \to R$, defined by $J(x) := y$, $z = x + iy \in H$, $R := \Re H$, the real part of H, being the real subspace of V. It is easily observed that J is well defined and $J^2 = -1$, hence $V = (V, R, J)$ is a CR vector space. The CR morphism $f : U \to V$ is naturally extended to a complex morphism $f^c : U^c \to V^c$, which induces a morphism $\mathrm{H}_U \to H_V$. These processes are reciprocal and hence they define categorically equivalent objects.

1.3 Categorical complexification

The (categorical) complexification is defined as follows:

Definition 1.1 Let X be an object in the category; then a pair (X^c, c) of an object X^c and a monomorphism $c : X \to X^c$ is a complexification of X, if for any pair (Y, g) of a (totally) complex object Y and a morphism $g : X \to Y$ there exists a unique morphism $h : X^c \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{c} & X^c \\ g\searrow & & \swarrow h \\ & Y & \end{array}$$

Remark 1.1 From the definition it follows immediately that a complexification is, if it exists, unique up to isomorphism; if there exist another complexification $c' : X \to X'$, then we have a morphism $h : X^c \to X'$ and a morphism $h' : X' \to X^c$ such that $hc = c'$ and $h'c' = c$. Then we have $hh'c' = c'$ and $h'hc = c$. From the uniqueness condition above we have $hh' = 1_{X'}$ and $h'h = 1^c_X$. Thus h, h' are the (categorical) isomorphisms.

1.4 CR real form of complex vector space

Here we will introduce the notion of CR real form of complex objects. First in the linear CR category $\mathcal{CR}$, let $V = (V, J)$ be a complex vector space; V is a $2n$-dimensional real vector space with an almost complex structure tensor J on V. Then we can give the following

Definition 1.2 A pair (V', c) of a CR vector space $V' = (V', R, J)$ of type (r, s) and a morphism $c : V' \to V$ of V' to V is a *codimension s or type (r, s) CR real form* of V, if $V = (V, c)$ is a categorical complexification of V', where $n = r + s$.

We have proved in the first paper [2] of a series of our works, that there exists always one and only one complexification U^c for any CR vector space U. In that paper we have not mentioned the converse problem; whether there exists always a CR real form or not (or the uniqueness of the real form). Here we give the answer to the problem; namely we have

Theorem 1.1 *There exists a CR real form of any possible type. Furthermore it is unique up to isomorphism; if there exists another codimension s CR real form (U, c') of V, then there exists an isomorphism $\iota : V' \to U$ such that the following diagram commutes:*

$$\begin{array}{ccc} V' & \xrightarrow{c} & V \\ \iota \searrow & & \nearrow c' \\ & U & \end{array}$$

Proof.
(i) Existence: First fix a desired type (r, s) of the CR real form V'. We can take, without loss of generality, a base $\{e_1, e_2, \cdots, e_{2n}\}$ of $V = (V, J)$ so that $Je_k = e_{r+k}$ for $k = 1, 2, \cdots, r$ and $Je_k = e_{s+k}$ for $k = 1, 2, \cdots, s$. We should note $r + s \leq n$ since we have $\dim V' = 2r + s \leq 2n = \dim V$ by assumption. Then the real vector space R spanned by the vectors $\{e_1, e_2, \cdots, e_{2r}\}$ forms a complex vector space $R = (R, J_R)$ with the induced complex structure $J_R := J|R$. Now take the real s-dimensional subspace S spanned by the elements $\{e_{r+1}, \cdots, e_{r+s}\}$. We have $R \cap S = \mathbf{0}$ and denote the direct sum $R \oplus S$ by V'. Then the CR vector space $V' = (V', R, J')$ with the complexification c defined by the canonical inclusion $c : V' \hookrightarrow V$ is the desired one.
(ii) Uniqueness: For another CR real form $U = (U, c')$ let us first define a morphism $\iota : V' \to U$ in the following way. For an element $v \in V'$ we have the description $v = \sum_{j=1}^{2r+s} v_j e_j$. According to the direct sum decomposition we have the direct summand of $c(v) = z + u$ by $z = \sum_{j=1}^{2r} v_j e_j \in R$ and $u = \sum_{j=2r+1}^{2r+s} v_j e_j \in S$. On the other hand for another CR real form, $c' : U \to V$ we have the corresponding direct sum decomposition $U = R_U \oplus S_U$, which induces the natural morphism $c'(v') = z + u$ for any element $z \in R$ and $u \in S$ as above. Thus we can define $\iota(v)$ by $\iota(v) := v'$, which, of course, satisfies the compatibility condition. Q.E.D.

Remark 1.2 It is verified that the complexification morphism is epic, which is, however, not explicitly stated anywhere in a series of my recent works [2, 3, 7, 8, 12, 13, 14, 15].

1.5 CR Lie algebra

As a subcategory of linear complex spaces, we have the category of complex Lie algebras, whose object is a complex Lie algebra $(L, [,])$ with the complex bilinear Lie bracket operation $[,]$. Then a morphism $f : L_1 \to L_2$ is a complex linear mapping of L_1 to L_2 which preserves the bracket operation. By replacing the adjective complex with CR we have the category of CR Lie algebras, whose integrated category is that of the CR Lie groups [3, 7]. This is not the full subcategory of the category of CR vector space, as is easily verified. Anyway we have already the complexification theorem [3] and here we consider the converse problem: the existence and uniqueness of CR real form in the CR Lie algebra category. Contrary to the case of CR vector space we cannot establish the corresponding theorem, as the following counter-example shows.

Example. Let $L = (L, [,], J)$ be a complex semi-simple Lie algebra, say $L = sl(2, \mathbf{C})$. Then L is a 3-dimensional complex vector space. For $r = 1, s = 1$, where $n = 3 = 2r + s$, if we have a CR real form L' of L, then $(1, 1)$ type CR structure $R = R_{L'}$ is an ideal of the CR Lie algebra L' [7]. Thus the corresponding complex Lie subgroup of $SL(2, \mathbf{C})$ is an invariant subgroup of dimension 1, which is impossible since it is simple and semi-simple. Eventually we can prove the following

Theorem 1.2 *In the category of CR Lie algebras there exists a type (r, s) CR real form uniquely up to isomorphism iff the complex Lie algebra $L = (L, J)$ has a complex ideal R_L of dimension r.*

Proof. It is a routine work (a slight modification of the classical result as is given in the remark below), and hence left to the readers.

Remark 1.3 For $r = 0$, type (r, s) CR real form is nothing but the usual real form in the $\mathbf{R}/\mathbf{C}$ Lie algebra theory. Then the condition is empty since R_L is trivial and it is well known that each (existing) real form is conjugate (isomorphic) via the inner automorphism.

2 Category of G-CR spaces

2.1 G-complex space

We generalize the definition of complex category in the linear case to that in nonlinear case. The integrated category of linear complex spaces is that of the (linear or non linear) complex spaces in the classical sense equipped with the coherent structure sheaf (of $\mathcal{O}$ modules). From the geometric point of view, it would be important whether the object is with/without singularities. From the categorical point of view, we could translate the problem of singularities to that of morphisms of a special nature. Nonsingular objects are complex manifolds, which is easily treated by using the local holomorphic coordinates. We could not, however, preserve these objects under lots of functorial operations, such as fiber product, *etc.,* or quotient. Thus we are forced to introduce the complex manifolds with singularities. Non singular objects are categorically equivalent to the smooth manifolds whose tangent bundles are equipped with the integrable almost complex structure tensor.

In this case morphisms are assumed to be smooth mappings preserving these structures. The next one is the category of complex Lie groups. It is the integration of the category of complex Lie algebras, in the sense that each object is an integration of some object in the latter one. Although the correspondence of integration is not injective, the converse one, *i.e., differentiation* which is defined in details elsewhere, is injective. Now we can consider the category of G-complex spaces, whose object is a triplet (G, X, ρ) of a complex Lie group G, a complex space X, and an (effective) holomorphic group action $\rho : G \times X \to X$. A morphism $f = (f_1, f_2) : (G, X, \rho) \to (G', X', \rho')$ is composed of the respective morphisms $f_1 : G \to G'$ and $f_2 : X \to X'$ satisfying the following commutative diagram:

$$\begin{array}{ccc} G \times X & \xrightarrow{f} & G' \times X' \\ \rho \downarrow & & \downarrow \rho' \\ X & \xrightarrow{f_2} & X' \end{array}$$

If both of the spaces X and X' reduce to the singletons $\{x\}$ and $\{x'\}$ respectively, we have the usual complex Lie group category. On the contrary if the groups G and G' reduce to the trivial groups respectively, we have the usual (non linear) complex category.

2.2 G-CR space

We generalize the definition of CR category in the linear case to that in linear/nonlinear case. This category contains the complex category as a (full) subcategory. The object is a (real analytic) CR manifold with/without singularities and the morphism is a (smooth) CR mapping. Now we can generalize the category of complex Lie groups to that of CR one; the latter one contains the former one as a (full) subcategory. It is also considered the integration of the category of CR Lie algebra, which is conversely composed of the left invariant CR vector fields on the corresponding (minimal model of) CR Lie group. This has never been treated systematically yet, as far as the author does know.
Then we can extend the category of G-complex spaces to G-CR ones, whose object is a triplet (G, X, ρ) of a CR Lie group G, a CR space X, and an (effective) CR group action $\rho : G \times X \to X$. A morphism $f = (f_1, f_2) : (G, X, \rho) \to (G', X', \rho')$ is composed of the respective morphisms $f_1 : G \to G'$ and $f_2 : X \to X'$ satisfying the same commutative diagram as above. Then this category is denoted $\mathcal{GCR}$ henceforth.

Example. Let M be the Euclidean n-space $(\mathbf{R}^n, e)$ endowed with the flat CR kähler structure (R, J, e) of type (r, s), where $n = 2r + s$, $n' = r + s$. Then we have $M_R^c = \mathbf{C}^n$ with the flat complex Euclidean metric e^c and $M_{CR}^c = \mathbf{C}^{n'}$ with the quotient flat Euclidean metric e_{CR}^c as were defined previously. Then we have the following commutative diagram:

$$\begin{array}{ccccc} (\mathbf{R}^n, e) & \xrightarrow{\iota_R} & (\mathbf{C}^n, e_R^c) & \xrightarrow{\pi_R} & (\mathbf{R}^n, e) \\ id \downarrow & & \downarrow \varpi & & \downarrow id \\ (\mathbf{R}^n, \mathbf{R}^{2r}, J, e) & \xrightarrow{\iota_{CR}} & (\mathbf{C}^{n'}, e_{CR}^c) & \xrightarrow{\pi_{CR}} & (\mathbf{R}^n, \mathbf{R}^{2r}, J, e) \end{array}$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & A = \ker p \\ & & & & \downarrow \\ 0 & \rightarrow & V_R = R \oplus S & \overset{c}{\rightarrow} & V_R^c = V \otimes_{\mathbf{R}} \mathbf{C} \\ & & id \downarrow & & \downarrow p \\ 0 & \rightarrow & V_{CR} = R \oplus S & \overset{c}{\rightarrow} & V_{CR}^c = V \otimes_{CR} \mathbf{C} \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

where p denotes the natural epimorphism in the CR category. Non flat ones will be given in [14, 15]. We have already the existence of the complexification in several categories [2, 7, 8, 12, 14, 15]. As was stated in the beginning, we are now to prove the existence of the categorical complexification in the GCR category.

Theorem 2.1 *There exists the complexification in $\mathcal{GCR}$.*

Proof. Before giving a proof in the general setting, let us review the complexification in some special cases.

The case 1)
We have constructed the complexification of the CR space using the tensor product operation in $\mathcal{CR}$ as was introduced in [15](cf [2]).
The case 2) .
We can apply this result to the case of CR Lie algebra, since the complexification of a CR Lie algebra L is obtained as a complexification of CR space automatically. In the integrated category of CR Lie algebra, *i.e.,* that of CR Lie group, the situation is a little bit more complicated. The correspondence of integration is not injective, as was classical in the celebrated *Fundamental Theorem of Analysis*, since the integral constants appear as ambiguity. To avoid this we rely on the topological consideration. First we take the universal covering complex Lie group G associated to the complexification of L. Then take its appropriate quotient that have the induced CR subgroup isomorphic to the original one (cf. [8]).
The case 3)
For a homogeneous space $X = G/H$ of a CR Lie group G with isotropy subgroup H, we have the categorical complexification using the method in the previous cases (cf. [8]).
The case 4)
For a (real analytic) CR manifold X with/without singularities, we have the well known local embeddability theorem (cf. *e.g.*[1]). So we can easily construct the germ of complexification X^c. Then we can take an appropriate geometric representation as a global object, which is realized as the adapted CR complexification, [14, 15]. This representation, however, depends apparently on the choice of a specific kähler metric at this moment.
The case 5)
The final case is in the general setting; we have a GCR object (G, X, ρ). There exist complexification G^c and X^c as independent objects in the respective category. We have to join them via the complexification $\rho^c : G^c \times X^c$ of ρ, so that the required diagram is well defined and commutes. This is done in the next section.

2.3 Orbit decompositions

Let (G, X, ρ) be a GCR object. If there exists the categorical complexification $c : G = (G, X, \rho) \to G^c = (G^c, X^c, \rho^c)$, we can apply the differentiation functor $\mathcal{D} : \mathcal{GCR} \to \mathcal{LCR}$ to this morphism c defined as follows: $\mathcal{D}(G) = TG, \mathcal{D}(X) = TX$ being the tangent bundles of G, X, respectively and $\mathcal{D}(\rho) = d\rho$, $\mathcal{D}(c) = dc$ are given by the usual differential of the respective smooth mapping ρ, c. Thus the candidates of the complexification in GCR category are given by the corresponding complexification in the LCR category, which is already established in the preceding cases. In other words we have the orbit decompositions of the CR Lie group action (G, ρ) on X, and on each orbit we have the complexification due to the results in [8]. The key point is the integrability (or compatibility) condition to paste the respective complexifications. The detailed description of the compatibility condition might be more easily understandable through the procedures in some basic examples given in the last section and possibly in [15].

3 Some examples and open problems

3.1 Basic examples

We give some basic examples in the following.

Example 1 Let $V = (V, J, R)$ be a CR vector space, and $G = GL(V)$ be a CR general linear group acting as the usual linear action ρ on V. Then the triplet (G, V, ρ) is a GCR object, whose categorical complexification is the triplet $(G^c_{CR}, V^c_{CR}, \rho^c_{CR})$ defined in [2] rather implicitly. Though the categorical property is not proved there explicitly, we can deduce it easily, which is hence left to the readers.

Example 2 Let $X = S^{2n-1}$ be a $(2n-1)$ dimensional unit sphere in the n-dimensional complex Euclidean space $\mathbf{C}^n$. Let G be any closed CR subgroup of the unitary group $U(n)$ acting as the usual linear action ρ on X. Then the triplet (G, X, ρ) is a GCR object, whose categorical complexification is the triplet (G^c, X^c, ρ^c), where G^c is the uniquely defined complexification of G in $U(n)^c$ and X^c is the adapted complexification of X in the CR category [13, 14]. The complex action $\rho^c : G^c \times X^c \to X^c$ is defined by the integration of the infinitesimal action of the CR Lie algebra $T_{CR}(G)$ of G on X.

3.2 Open problems

We can consider the complexification defined in the preceding section as a (covariant) functor from the GCR category to the GC (*i.e.*G-complex) one in the sense as follows [11]:
Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of X to Y and Y to Z, respectively in the GCR category. Then we have the complexification of respective objects and morphisms satisfying the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{g \circ f} & Z \\ f \searrow & & \nearrow g \\ & Y & \end{array}$$

$$\begin{array}{ccc} X^c & \overset{(g\circ f)^c}{\longrightarrow} & Z^c \\ f^c \searrow & & \nearrow g^c \\ & Y^c & \end{array}$$

So there arises a natural question whether this functor is invertible or not. If it is the case, for a GC object $G = (G, X, \rho)$ we have a GCR one $G^r = (G^r, X^r, \rho^r)$ such that $(G^r)^c = G, ((G^r)^c)^r = G^r$. This remains open at least in the most general setting, in the sense that there exists a counter example even in the differential category as has been pointed out in section 1.5. So the problem would be well-posed if we should ask suitable (necessary and sufficient) conditions for a GC object to have a CR real form.

4 Appendix

For the readers' convenience we give here basic terminologies in the category theory, some of them being applied to $\mathcal{CR}$.

4.1 Identity and isomorphism

Some basic morphisms are defined as follows. An endomorphism $1 = 1_M : M \hookleftarrow$ of an object M is an *identity* if $1g = g$, $h1 = h$ for any g, h if the compositions are defined. In $\mathcal{CR}$ we see easily that an endomorphism $f : M \hookleftarrow$ is an identity iff $f : M \hookleftarrow$ is an identity in $\mathcal{R}$. A morphism $f : M \to N$ is a *coretraction (or retraction)* if there exists a morphism $g : N \to M$ such that $gf = 1_M$ (or such that $fg = 1_N$). A morphism $f : M \to N$, which is at once a retraction and a coretraction is an *isomorphism*.

4.2 Images and coimages

Definition 4.1 ((co)image) For a morphism $f : U \to V$ a subobject $g : I \to V$ of V is an *image* of f and denoted $I = \mathrm{Im}(f)$ or $f(U)$ if
(i) f factors through I; $f = gh$ for some $h : U \to I$,
(ii) (universality condition) it is the minimal in the sense if $g' : I' \to V$ is another one with the property (i), there exists a morphism $g'' : I \to I'$ such that g factors through I'. The image $I = f(U)$ is said to be *epimorphic* if the canonical morphism $h : U \to I$ in the above is an epimorphism. Dualizing all above we have the notion of coimage of f: for $f : U \to V$ a quotient object $g : U \to Q$ of U is a *coimage* of f and denoted $Q = \mathrm{Coim}(f)$ if
(i') f factors through Q; $f = \pi g$ for some $\pi : Q \to V$,
(ii') it is the maximal in the sense if $g' : U \to Q'$ is another one with the property (i'), there exists a morphism $g'' : Q' \to Q$ such that g factors through Q'.

Remark 4.1 (Uniqueness of (co)image) From the universality condition (ii), or (ii') above we derive the uniqueness of the image I and coimage Q (up to isomorphism), which we might be justified to denote Im (f) (or Coim (f)). Hence the uniqueness of h (or π), too.

Then we have

Theorem 4.1 ([2, 13]) *The category $\mathcal{CR}$ has epimorphic images.*

Proof.
(1) Let $f : U \to V$ be any morphism. Then as a candidate of the image we take $I =$

$(I, R_I, J_I) := (f(U), f(R_U), J_V|f(R_U))$, where $g : I \to V$ is the set-theoretic canonical inclusion $I \hookrightarrow V$, $h : U \to I$ is to be naturally induced from f itself. We have to show these are well defined object and morphisms of $\mathcal{CR}$ with the required property. First we have the following canonical commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
R_U & \overset{h}{\to} & R_I & \overset{g}{\to} & R_V \\
\downarrow j_U & & \downarrow j_I & & \downarrow j_V \\
U & \overset{h}{\to} & I & \overset{g}{\to} & V
\end{array}$$

whose columns are exact. The commutativity is shown by the standard arguments. The crucial point is the exactness of the central column, which follows from the commutativity of the diagram below:

$$\begin{array}{ccccc}
R_U & \overset{h}{\to} & R_I & \overset{g}{\to} & R_V \\
\downarrow J_U & & \downarrow J_I & & \downarrow J_V \\
R_U & \overset{h}{\to} & R_I & \overset{g}{\to} & R_V
\end{array}$$

(2) Now we are to proceed to the next step, i.e. the proof of the universality condition (ii). Let $g' : I' \to V$ be another one with the above property (i). Since we can assume $f(U)$ to be the image of f in $\mathcal{R}$, and g' to be a morphism in $\mathcal{R}$, we have the $\mathbf{R}$ linear mapping $g'' : I \to I'$ such that the diagram below commutes:

$$\begin{array}{ccc}
I & \overset{g''}{\longrightarrow} & I' \\
g \searrow & & \swarrow g' \\
& V &
\end{array}$$

There remains to prove g'' defines a morphism which is rather a routine work hence omitted.
(3) The last step is to prove $h : U \to I$ is epimorphic, which is immediate from the definition. Q.E.D.

4.3 Monomorphisms and subobjects

Now we characterize monomorphisms of $\mathcal{CR}$ in set-theoretic terms. Let $f : U \to V$ be a CR morphism and $I = (f(U), g)$ be the categorical image of f, with the canonical inclusion $g : f(U) \hookrightarrow V$. As was remarked in [13], g is monic and hence

Theorem 4.2 *The categorical image I is a subobject of V.*

Remark 4.2 It should be noted that a subset V in a CR space $U = (U, R_U, J_U)$ may carry various CR structures (R_V, J_V) with which V is considered a CR subobject $j : V \hookrightarrow U$ of U, where j is the canonical inclusion. Then the set

$$\mathrm{CR}(V) := \{(R_V, J_V) | V = (V, R_V, J_V) \text{ is a subobject of } U\}$$

is partially ordered with the canonical inclusion order. Thus the set-theoretic identity 1_V defines various possible morphism $\iota_{1,2} : V_1 = (V, R_1, J_1) \to V_2 = (V, R_2, J_2)$ according to this partial order.

References

[1] Boggess,A., *CR manifolds and the tangential Cauchy-Riemann complex*, Studies in Adv. Math. CRC Press,1991.

[2] Iwata,K.,Shiga,K. and Takeuchi,S., Category of CR vector spaces, Sc. Rep. Fac. Educ., Gifu Univ.**17**(1993),1-18.

[3] Iwata,K., and Takeuchi,S., Category of CR algebras and CR Lie algebras, Sc. Rep. Fac. Educ., Gifu Univ. **18-2**(1994),55-73.

[4] Kobayashi,S., Submersions of CR submanifolds, Tohoku Math.J.**89**(1987),95-100.

[5] Lemperet,L. and Szöke,R., Global solutions of the homogeneous complex Monge-Ampère equations and complex structure on the tangent bundle of riemannian manifolds, Math.Ann.290(1991),689-712.

[6] Mitchell,B., *Theory of categories*, Academic Press, New York and London, 1965.

[7] Shiga,K.,Takeuchi,S., CR Lie groups and invariant CR structures on CR Lie transformation groups, Sci. Rep. Fac. Educ., Gifu Univ.**18-1**(1993), 5-15.

[8] Shiga,K.,Takeuchi,S., Categorical complexification of CR Lie algebras and CR Lie groups, Sci. Rep. Fac. Educ., Gifu Univ.**19-1**(1994),5-15.

[9] Shiga,K.,Takeuchi,S., Equivariant complexification of homogeneous space of CR Lie group, Sci. Rep. Fac. Educ., Gifu Univ.**20-1**(1995),1-24.

[10] Shiga,K.,Takeuchi,S., CR tensor product, Sci. Rep. Fac. Educ., Gifu Univ.**21-2**(1997),13-19.

[11] Takeuchi,S., Complexification as a functor from $\mathcal{CR}$ to $\mathcal{C}$, Collected papers on several complex variables edited by J.Kajiwara, Seminar on several complex variables, Graduate School of Mathematics, Kyusyu University, (1994).

[12] Takeuchi,S., Categorical images of CR morphisms, Sc. Rep. Fac. Educ., Gifu Univ.**25-2**(2001),1-5,

[13] Takeuchi,S., On the adapted complexification of CR manifold, Sc. Rep. Fac. Educ., Gifu Univ.**28-1**(2003),1-9.

[14] Takeuchi,S., On the categorical complexification of CR Lie group action, Proceedings of the 11th International Conference on Finite and Infinite Dimensional Complex Analysis and related topics, Chiang Mai University, July 27-31, 2003.

[15] Takeuchi,S., Some remarks on the complexification in the GCR category, in preparation.