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On the adapted complexification of CR manifold [1] [2]

by
Shigeru Takeuchi [3]

Department of Mathematics, Faculty of Education, Gifu University
Yanagido 1-1, Gifu, Japan
email: takeuchi@cc.gifu-u.ac.jp
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The purpose of this paper is to give a notion of the adapted complex structure on some quotient vector bundle of the tangent bundle of real analytic complete CR kählermanifolds: to formulate the complexification and prove the existence and the uniqueness. We refer the readers to [7, 9, 13, 21, 22] for more details of the (finite dimensional) CR category as basic ingredients.

1 CR category and complex category

1.1 Linear CR category

First let us recall, for the readers' convenience, a *CR vector space* is a triplet (V, R, J) of a real vector space V, a subspace R of V and a complex structure J on R. An $\mathbf{R}$-linear mapping $f : V_1 \to V_2$ of a CR space $V_1 = (V_1, R_1, J_1)$ to another one $V_2 = (V_2, R_2, J_2)$ is a *(CR) morphism* if f preserves the CR structures R_i, *i.e.*, $f(R_1) \subset R_2$ and f commutes the complex structures J_i, *i.e.*, $fJ_1 = J_2 f$. Let U, V and W be CR spaces. An $\mathbf{R}$-bilinear mapping $f : U \times V \to W$ is a *CR bilinear morphism* if f induces the CR morphisms to W for each fixed element of the respective factors U, V. (We omit the adjective CR if there is no fear of confusion.) As will be introduced in a later section, the tangent spaces of CR manifolds are prototypes of CR spaces and the differentials of CR mappings are the CR morphisms of these tangent spaces.

1.2 Complex representation of CR vector space

Letting $i = \sqrt{-1}$, we have the following complex representationa for a CR vector space $V = (V, R, J)$; the usual complexification $V^c := V \otimes_{\mathbf{C}}$ of V, considered as a real vector space, has the direct sum splitting

$$V^c := V \otimes_{\mathbf{C}} = (R \oplus S)^c = H \oplus A \oplus S^c,$$

where S is a complementary direct summand of R in V with $H = H_V = \{x - iJx \in R^c | x \in R\}$ and $A = A_V = \overline{H} = \{x + iJx | x \in R^c\}$ being the $\pm i$ eigenspace of J respectively. We call $(A)H$ the (anti) holomorphicsubspace of R^c, respectively. Conversely if we are given a direct sum splitting $H \oplus \overline{H}$ of a complex subspace of a complexified

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vector space V^c of V, then we have a unique real endomorphism $J : R \to R$, defined by $J(x) := y$, $z = x + iy \in H$, $R := \Re H$, the real part of H, being the real subspace of V. It is easily observed that J is well defined and $J^2 = -1$, hence $V = (V, R, J)$ is a CR vector space. The CR morphism $f : U \to V$ is naturally extended to a complex morphism $f^c : U^c \to V^c$, which induces a morphism $\mathrm{H}_U \to H_V$. These processes are reciprocal and hence they define categorically equivalent objects.

1.3 CR tensor product

Here we will briefly review the tensor product of CR spaces. Before recalling it, we prepare a few notational conventions. Let U, V and W be CR vector spaces. Then considering them as $\mathbf{R}$-spaces, we have the usual tensor product $T_R = U \otimes_R V$ and the canonical $\mathbf{R}$-bilinear mapping $\iota' : U \times V \to T_R$, such that there exists uniquely an $\mathbf{R}$-linear morphism $\phi' : T_R \to W$ for any $\mathbf{R}$-bilinear mapping $g : U \times V \to W$ satisfying $g = \phi' \circ \iota'$. Then take a subspace $I = I(U, V)$:

$$I = \langle (J_U u) \otimes v - u \otimes (J_V v); u \in R_U, v \in R_V \rangle_R$$

of T_R. We put $K_g := \{(u, v) \in U \times V; g(u, v) = 0\}$, the *zero* (locus) of g. Then since $I \subset \langle \iota'(K_g) \rangle = \ker \phi'$. we can take the quotient space $T := T_R / I(U, V)$ and thus we have

Theorem 1. 1 ([7]) *Let U, V be CR spaces . Then there exist a CR space T and a CR bilinear morphism $\iota : U \times V \to T$ satisfying the following conditions:*

T1 *For any CR space W and a CR bilinear morphism $g : U \times V \to W$, there exists a morphism $\phi : T \to W$ such that $\iota \circ \phi = g$.*

T2 *The pair (ι, T) is determined uniquely up to CR isomorphism; if there exists another space T' and a morphism $\iota' : U \times V \to T'$ with the above property, then there exists uniquely a CR isomorphism $\varphi : T \to T'$ such that $\iota \circ \varphi = \iota'$,*

Under the above situation we give

Definition 1. 1 The pair (T, ι) of T and the CR bilinear morphism ι or simply the CR space T is the *CR tensor product* of U, V and is denoted $U \otimes_{CR} V$.

1.4 Hermitian CR vector space

Now we consider the following setup. Let $V = (V, R, J)$ be a CR vector space. Then an inner product $g = (\cdot, \cdot)$ on V is *(CR) hermitian* if the CR structure J preserves this inner product on R; $(Ju, Jv) = (u, v)$ for $\forall u, v \in R$. Then we give the following

Definition 1. 2 A *hermitian CR vector space* is a pair $V = (V, R, J, g)$ of a CR vector space $V = (V, R, J)$ and a CR hermitian inner product $g = (\cdot, \cdot)$.

Let $\mathcal{CR}$ (resp. $\mathcal{C}$) be the category of CR (resp. C) objects $V = (V, R, J)$ (resp. $V = (V, J)$), where morphism $f : U \to V$ is defined to be a mapping which preserves the CR structure and commutes J. If it is appropriate to distinguish the three objects $V \in \mathcal{R}$, $V \in \mathcal{CR}$, $V \in \mathcal{C}$, we denote them V_{R}, V_{CR}, and V_{C} respectively. First let us consider CR vector space $V = (V, R, J)$ forgetting the hermitian inner product and the CR structure

(R, J) of V we have the totally real object $V_{\mathrm{R}} := (V, 0)$, and the usual complexification $c : V \to V_{\mathrm{R}}^c := V \otimes_{\mathrm{R}} \mathbf{C}$. As for the inner product $g = g_{\mathrm{R}}$ we have the natural tensor product extension $g_{\mathrm{R}}^c : V_{\mathrm{R}}^c \times V_{\mathrm{R}}^c \to \mathbf{C}$ as usual. On the other hand we have introduced the notion of the CR tensor product of a CR vector space $V = (V, R, J)$ forgetting the hermitian inner product [20]. Then associated to V is the CR complexification $c : V \to V_{\mathrm{CR}}^c := V \otimes_{\mathrm{CR}} \mathbf{C}$. As usual we can take the orthogonal complement $R^\perp$ of R w.r.t the metric g. Then we have the direct sum splitting $V = R \oplus R^\perp$ and $V_{\mathrm{CR}}^c = R \oplus (R^\perp)^c$, where $(R^\perp)^c$ denotes the usual complexification of $R^\perp$. Then we have

Lemma 1. 1 *The CR hermitian inner product $g = g_{\mathrm{CR}}$ has the CR tensor product extension $g_{\mathrm{CR}}^c : V_{\mathrm{CR}}^c \times V_{\mathrm{CR}}^c \to \mathbf{C}$ so that the following diagram commutes:*

$$\begin{array}{ccccc}
V \times V & \xrightarrow{g} & \mathbf{R} & & \\
 & \searrow & & \searrow & \\
\downarrow & & V_{\mathrm{CR}}^c \times V_{\mathrm{CR}}^c & \xrightarrow{g_{\mathrm{CR}}^c} & \mathbf{C} \\
 & \swarrow & & \swarrow & \\
V_{\mathrm{R}}^c \times V_{\mathrm{R}}^c & \xrightarrow{g_{\mathrm{R}}^c} & \mathbf{C} & &
\end{array}$$

Proof. Let us define the CR extension $g_{\mathrm{CR}}^c : V_{\mathrm{CR}}^c \times V_{\mathrm{CR}}^c \to \mathbf{C}$ by the following:

$$g_{\mathrm{CR}}^c(z + x + iy, w + u + iv) := h(z, w) + h(x, u) - h(y, u) + ih(x, v) + h(y, u)$$

where $z, w \in R\, x, y, u, v \in R^\perp$. It is easily verified that the above diagram is commutative. Q.E.D.

1.5 Complexifications

Let us consider the *categorical complexification* in $\mathcal{CR}$ defined as follows:

Definition 1. 3 Let X be an object in the category. Then a pair (X^c, c) of an object X^c and a *monomorphism* $c : X \to X^c$ is a *complexification of* X, if for any pair (Y, g) of a (totally) complex object Y and a morphism $h : X \to Y$ there exists a unique morphism $h : X^c \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{c} & X^c \\
g \searrow & & \swarrow h \\
 & Y &
\end{array}$$

Remark 1. 1 ¿From the definition it follows immediately that a complexificationis, if it exists, unique up to isomorphisms; if there exist another complexification$c' : X \to X'$, then we have a morphism $h : X^c \to X'$ and a morphism $h' : X' \to X^c$ such that $hc = c'$ and $h'c' = c$. Then we have $hh'c' = c'$ and $h'hc = c$. ¿From the uniqueness condition above we have $hh' = 1_{X'}$ and $h'h = 1_X^c$. Thus h, h' are the (categorical) isomorphisms, which are mutually inverse.

Theorem 1. 2 ([7, 18]) *The categorical complexification does exist in the linear CR category.*

For the nonlinear cases we have partial results as in [18, 19, 23, 24]. In the next section we introduce a complexification, that induces the categorical complexification through the functor $\mathcal{L}$ from the nonlinear category to the linear one. The functor $\mathcal{L}$ is defined by taking the differential of the objects and the morphisms.We will investigate the fundamental structure of a complexification, if it is realized as the adapted complexification in the CR category.

2 Hermitian CR manifold

2.1 Hermitian CR structure

Let $M = (M, g)$ be a (real analytic) complete riemaniann manifold with a riemannian metric g. We denote the tangent space at $p \in M$ by T_pM, the tangent bundle by TM and the (germ of local) smooth sections by $\Gamma(TM)$. A complex structure$J$ (of M) means by abusing the terminology an integrable almost complex structure tensor J (on M), where *integrable* means, due to the celebrated theorem of Nirenberg [14], the vanishing of Nijenhuis tensor:

$$S(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],\ X, Y \in \Gamma(TM)$$

First we give the following

Definition 2. 1 A quadruplet $M = (M, g, R, J)$ is a *hermitian CR manifold*, if $M = (M, g)$ is a riemaniann manifold, and $R = R_M$ is a smooth subbundle of TM and $J = J_M$ is a complex structure on R that preserves g:

$$g_p(Ju, Jv) = g_p(u, v),\ \forall p \in M,\ u, v \in R_pM$$

.

Further if the CR structure (R, J) is *compatible* with the metric g in the sense that the *CR-fundamental 2-form* ω defined as below (*) is *CR-closed*, we say that the quadruplet M is a CR-kähler manifold.

$$(*) \quad \omega(X, Y) = g(X, JY),\ X \in \Gamma(TM), Y \in \Gamma(R_M)$$

Remark 2. 1 We note that if $R = TM$, M is *totally complex* in our terminology. Then we have the usual hermitian /kählermanifold $M = (M, g, J)$. The opposite extreme is the self-evident totally real hermitian manifold $M = (M, g)$, i.e., $R = 0$ case, a little bit different terminology as usual in differential geometry. It should be noted that any riemannian manifold is a totally real hermitianmanifold and hence automatically kähler since the condition is vacant.

To state what implies the kählercondition we need the following terminology.

Definition 2. 2 A (local) smooth function u on M is *CR-strongly pluri-subharmonic* (for short *spsh* henceforth) if the Levi form $\mathcal{L}u$ of u defined as follows is positive definite on R_M^c:

$$\mathcal{L}u := (Z_i \overline{Z}_j u)_{i,j}$$

where $\{Z_i | i = 1, \cdots r\}$/ $\{\overline{Z}_j | j = 1, \cdots r\}$ denote complex bases for the complex vector space H_M/ A_M, respectively.

As a generalization of the complex case, we can prove the following

Proposition 2. 1 *The kähler condition on g implies that it is induced locally from a CR-spsh function u on a neighbourhood of each point of M:*

$$g(Z_i, \overline{Z}_j) = \mathcal{L}u_{i,j}.$$

We'll give a detailed proof in [24], where some examples are to be shown.

2.2 Adapted complex structure

Let (M, g) be a real analytic complete riemannian manifold and take a tubular neighbourhood of M in the tangent bundle TM as

$$T^{\delta}M := \{v \in TM | \sqrt{g(v,v)} < \delta\}$$

for a positive real number δ. Then we have the following

Definition 2. 3 (adapted complex structure [12]) A complex structure J on the tangent bundle $T(T^{r}M)$ of $T^{r}M$ is said to be *adapted* if for any geodesic $\gamma : \mathbf{R} \to M$ through a point $p \in M$, the associated mapping $\tilde{\gamma} : \mathbf{C} \to TM$ defined by

$$\tilde{\gamma}(z) := (t\dot{\gamma}(s))$$

is holomorphic, where $z = s + it \in \mathbf{C}$ and $\dot{\gamma}(s)$ denotes the tangent vector $d\gamma/ds$ to γ at $q = \gamma(s)$.

Then it is known [4, 15, 16] that the adapted complex structure does exist locally in general and globally for the cases of compact manifolds and homogeneous spaces in *e.g.*,[12, 17]. In general the possible canonical complex structure is given by the natural splitting of TM into the sum of the horizontal and the vertical component except for the integrability condition, which is expressed in terms of connections on M. This is the essential content of the classical works of Sasaki [15] and Dombrowski [4], which state the (necesarry and) sufficient condition to be the vanishing of torsion and curvature tensor. The typical cases are flat ones, *i.e.*the usual euclidean space $\mathbf{R}^n$. Combining this with the classical embedding theorem of Grauert we have the following well known

Theorem 2. 1 ([5]) *There exists a complexification $\iota : M \to M^c$ of a real analytic complete riemannian manifold M which satisfies the following conditions:*
(i) *The complexification (ι, M^c) is realized by the zero section $\iota : M \to TM$ and some of its tubular neighbourhoods $T^{\delta}M$.*
(ii) *The complex structure J on the tangent bundle $T(M^c)$ of M^c is given by the following formula; for any geodesic $\gamma : \mathbf{R}$ through a point $p \in M$, the associated mapping $\tilde{\gamma} : \mathbf{C} \to TM$ defined by*

$$\tilde{\gamma}(z) = t\dot{\gamma}(s)$$

is holomorphic.
(iii) *There exists (uniquely) a kähler metric h on M^c, which induces the riemannian metric g on M; $\iota^* h = g$.*
(iv) *There exists a (smooth) deformation retraction $\rho : M^c \to M$, which is realized as the restriction of the canonical projection $\pi : TM \to M$.*
(v) *There exists a spsh function ϕ on M^c_{R} such that the kähler metric h is defined by the Levi metric of ϕ as in Proposition 2.1.*

2.3 adapted CR structure

We want to generalize the result in the previous section to CR cases. Categorically speaking the CR objects are considered real ones by the natural forgetful functor. Hence every hermitian CR manifold has the adapted complex structure by the above theorem. To be more precise let us take the tangent bundle TM of M. Then it gives the adapted complexification $\iota_{\mathrm{R}} : M_{\mathrm{R}} \to M_{\mathrm{R}}^{\mathrm{c}}$ of a real analytic complete riemannian manifold $M_{\mathrm{R}} = (M, g)$. On the other hand if we consider the metric CR vector space $T_p := (T_pM, R_pM, J_p, g_p)$ we can take the quotient CR vector space $Q_p := T_pM/R_pM$ with the trivial CR structure and the induced metric, denoted also g_p; we can take the orthogonal complement $R^{\perp} = R_p^{\perp}M$ of $R = R_pM$ in T_pM w.r.t. the hermitian metric g. Since Q_p is isomorphic to $R_p^{\perp}$ as a real vector space, we have the induced metric g_p on Q_p by identifying Q_p with $R_p^{\perp}$ and by restricting g_p to $R^{\perp}$. By collecting these metric real vector spaces (Q_p, g_p) we have the (riemannian quotient) vector bundle $QM = \cup_{p\in M} Q_p$ with the (induced riemannian) metric $g = \cup_{p\in M} g_p$ on QM. We denote by ϖ the quotient map $TM \to QM$. Let us denote a tubular neighbourhood of the zero sction of QM by $Q^{\delta}M$;

$$Q^{\delta}M := \{v \in QM | \sqrt{g(v,v)} \leq \delta\}.$$

We can define a quotient complex structure $\tilde{J}$ on the tangent bundle $Q^{\delta}M$ under the follwoing definition:

Definition 2. 4 A CR structure (R, J) on a hermitian CR manifold $M = (M, R, J, g)$ is adapted if the adapted complex structure $J : \mathrm{T}(\mathrm{T}^{\delta}M) \hookleftarrow$ preserves the tangent bundle TR of the total space of the CR structure $R = R_M$:

$$J(TR_M) = TR_M,$$

Then M is called an adapted CR manifold.

We have the following generalization of **Theorem 2.1**, the notations being the same as before. We give here only the statement of, and a few remarks on, the theorem. For a detailed description of $M_{\mathrm{CR}}^{\mathrm{c}}$ and its metric property, *e.g.*,Kählerness see the forthcoming papers [23, 24], where we discuss the adapted complexificationis categorical or not.

Theorem 2. 2 (adapted complexification) *Let $M_{\mathrm{CR}} = (M, R, J, g)$ be a real analytic complete adapted CR kähler manifold. Let $M_{\mathrm{R}} = (M, g)$ be the complete riemannian manifold, which is obtained from M_{CR} deprived of the additional CR structure. Then there exist a CR complexification $\iota_{\mathrm{CR}} : M_{\mathrm{CR}} \to M_{\mathrm{CR}}^{\mathrm{c}}$ of M_{CR}, a (usual) complexification $\iota_{\mathrm{R}} : M_{\mathrm{R}} \to M_{\mathrm{R}}^{\mathrm{c}}$ of M_{R} and a holomorphic submersion $\varpi : M_{\mathrm{R}}^{\mathrm{c}} \to M_{\mathrm{CR}}^{\mathrm{c}}$ which satisfy the following conditions:*
(i) *The complexification $(\iota_{\mathrm{CR}}, M_{\mathrm{CR}}^{\mathrm{c}})$ is realized by the zero section $\iota : M_{\mathrm{CR}} \to QM$ and its suitable tubular neighbourhood $Q^{\delta}M$,*
(ii) *The complex structure J on the tangent bundle $T(M_{\mathrm{CR}}^{\mathrm{c}})$of $M_{\mathrm{CR}}^{\mathrm{c}}$ is given by the following formula; for any geodesic $\gamma : \mathbf{R} \to M$ through a point $p \in M$, the associated mapping $\varpi\tilde{\gamma} : \mathbf{C} \to QM$ defined by*

$$\varpi\tilde{\gamma}(z) = \varpi(t\dot{\gamma}(s))$$

is holomorphic.
(iii) *There exists (uniquely) a kähler metric h on $M_{\mathrm{CR}}^{\mathrm{c}}$, which induces the (CR) kähler*

metric g on M_{CR}; $\iota_{\mathrm{CR}}^{*}h = g$.
(iv) *The canonical projection $\pi_{\mathrm{CR}} : M_{\mathrm{CR}}^{c} \to M_{\mathrm{CR}}$ is a CR deformation retraction, which makes the following diagram commutative:*

$$\begin{array}{ccccc} M_{\mathrm{R}} & \overset{\iota_{\mathrm{R}}}{\rightarrow} & M_{\mathrm{R}}^{c} & \overset{\pi_{\mathrm{R}}}{\rightarrow} & M_{\mathrm{R}} \\ id \downarrow & & \downarrow \varpi & & \downarrow id \\ M_{\mathrm{CR}} & \overset{\iota_{\mathrm{CR}}}{\rightarrow} & M_{\mathrm{CR}}^{c} & \overset{\pi_{\mathrm{CR}}}{\rightarrow} & M_{\mathrm{CR}} \end{array}$$

(v) *There exists a spsh function φ on M_{CR}^{c} such that the kähler metric h is defined by the Levi metric of φ as in Proposition 2.1.*

Remark 2. 2 The above diagram is a *CR diagram* in the sense every arrow is a CR morphism.

Remark 2. 3 We can prove the integrability of the adapted complexification, since it is defined as the quotient complex structure of the original adapted complex structure of M, which is integrable. See [23, 24] for details.

Remark 2. 4 As for the expression of this bundle we have another interpretation. Let us take a CR tensor product $(T_pM_{\mathrm{CR}})^c := T_pM \otimes_{\mathrm{CR}} \mathbf{C}$. We have then the collection $\cup_{p\in M}(T_pM_{\mathrm{CR}})^c$ of these complex vector spaces ,which is denoted $(TM_{\mathrm{CR}})^c$ and referred to the *CR complexification* of the tangent bundle TM_{CR} of CR manifold M_{CR}. It is shown that $(TM_{\mathrm{CR}})^c$ carries the natural *complex CR vector bundle* structure on M. On the other hand we can identify the restriction of the tangent bundle $T(M_{\mathrm{CR}}^{c})$ of the adapted complexification M_{CR}^{c} of M_{CR} to M_{CR} with the CR complexification $(TM_{\mathrm{CR}})^c$ of the tangent bundle TM of M_{CR}: $T(M_{\mathrm{CR}}^{c})|_{M_{\mathrm{CR}}} \cong (TM_{\mathrm{CR}})^c$. This situation is illustrated in the following commutative diagram:

$$\begin{array}{ccc} T(M_{\mathrm{CR}}^{c}) & \overset{bdle-iso}{\cong} & (TM_{\mathrm{CR}})^c = T_{\mathrm{CR}}M \otimes_{\mathrm{CR}} \mathbf{C} \\ \pi_{\mathrm{Q}} \downarrow & \searrow & \downarrow \pi_{\mathrm{CR}}^{c} \\ M_{\mathrm{CR}}^{c} = Q^{\delta}M & \overset{\pi_{\mathrm{CR}}}{\rightarrow} & M_{\mathrm{CR}} \end{array}$$

3 Flat example

We give only an elementary flat example as follows:
Example. Let M be the euclidean n-space $(\mathbf{R}^n, e)$ endowed with the flat CR kähler structure (R, J, e) of type (r, s), where $n = 2r + s$, $n' = r + s$. Then we have the isomorphism $M_{\mathrm{R}}^{c} = \mathbf{C}^n$ with the flat complex euclidean metric e^c and $M_{\mathrm{CR}}^{c} = \mathbf{C}^{n'}$ with the quotient/induced flat euclidean metric e_{CR}^{c} as were defined in the previous section. Then we have the following commutative diagram:

$$\begin{array}{ccccc} (\mathbf{R}^n, e) & \overset{\iota_{\mathrm{R}}}{\rightarrow} & (\mathbf{C}^n, e_{\mathrm{R}}^{c}) & \overset{\pi_{\mathrm{R}}}{\rightarrow} & (\mathbf{R}^n, e) \\ id \downarrow & & \downarrow \varpi & & \downarrow id \\ (\mathbf{R}^n, \mathbf{R}^{2r}, J, e) & \overset{\iota_{\mathrm{CR}}}{\rightarrow} & (\mathbf{C}^{n'}, e_{\mathrm{CR}}^{c}) & \overset{\pi_{\mathrm{CR}}}{\rightarrow} & (\mathbf{R}^n, \mathbf{R}^{2r}, J, e) \end{array}$$

Non flat ones will be given in [23, 24].

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