岐阜大学教育学部研究報告（自然科学）第47巻, 1-5, 2023年３月
Sci. Rep. Fac. Educ. Gifu Univ. (Nat. Sci.) 47, 1-5, March 2023

The symmetric ribbon number of knots with symmetric union presentations

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Key words: symmetrc union, Jones polynomial, ribbon knot.
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Abstract. In this paper, we introduce the symmetric ribbon number for symmetric unions and show that there exists an infinite family of symmetric unions with symmetric ribbon number two by the Jones polynomial.

1 Introduction

A *symmetric union* [4] is a knot which is obtained from the connected sum of a knot and its mirror image by inserting some vertical twists along the symmetry axis to the diagram. A symmetric union is known to be a ribbon knot. In this paper, we define the *symmetric ribbon number* for a symmetric union which is closely related to the ribbon number of a ribbon knot [5] and study it by using the Jones polynomial.

For a polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$, *max-deg*$f(t)$ (or *min-deg*$f(t)$) is the maximal degree (or minimal degree) of $f(t)$ and let *red-deg*$f(t)$=*max-deg*$f(t)-$*min-deg*$f(t)$.

Theorem 1.1. *Let $\bar{K}$ be a knot with a 1-banded symmetric union presentation $D_K \cup D_K^*(n)$. Then we have $V_{\bar{K}}(t) = 1-(-1)^n t^{-n} + (-1)^n t^{-n} V_{K\sharp K^*}(t)$. In particular, if $V_{\bar{K}}(t) \neq 1$, then red-deg$V_{\bar{K}}(t) = 2$max-deg$V_{K\sharp K^*}(t)$.*

Theorem 1.2. *Let Let $\bar{K}$ be a knot with a 2-banded symmetric union presentation $D_K \cup D_K^*(m,n)$. If $D_K \cup D_K^*(0,n)$ and $D_K \cup D_K^*(m,0)$ are 1-banded symmetric unions then we have $V_{\bar{K}}(t) = 1-(-1)^{m+n} t^{-m-n} + (-1)^{m+n} t^{-m-n} V_{K\sharp K^*}(t)$. In particular, if $V_{\bar{K}}(t) \neq 1$, then red-deg$V_{\bar{K}}(t) = 2$max-deg$V_{K\sharp K^*}(t)$.*

The notation for prime knots up to 10 crossings is due to Rolfsen's book [1]. In Section 2, we shall define a symmetric union. In section 3, we shall define a *banded symmetric union presentation* for a symmetric union. In section 4, we shall prove Theorems 1.1 and 1.2, define the symmetric ribbon number for a symmetric union and show the existence of a symmetric union with symmetric ribbon number two.

2 Definitions

Definition 2.1. Let D be an unoriented link diagram in the 2-sphere. The *bracket polynomial* of D, $< D >$ can be defined as a polynomial which satisfies the following identities.

i) $< \bigcirc >= 1$,

ii) $< D \cup \bigcirc >= -(A^2 + A^{-2}) < D >$,

iii) $<$  $>= A <$  $> + A^{-1} <$  $>$.

Then we define the *Jones polynomial* of an oriented link L by $V_L(t) = \{(-A^3)^{-w(D_K)} < D_L >\}_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ for any diagram D_L for L, where $w(D_L)$ be the writhe of D_L [2].

Here we define a symmetirc union [4] as follows. We denote the tangles made of half twists by integers $n \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ as in Figure 1.

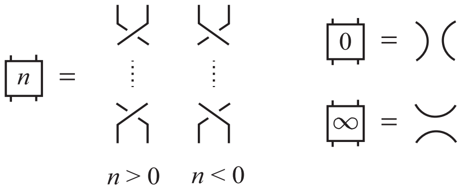

Figure 1: Tangles.

Definition 2.2. Let D_K be an unoriented diagram of knot K and D_K^* the diagram D_K reflected at an axis in the plane. We take k 0-tangles $T_i (i = 0, \ldots, k)$ on the symmetry axis as in Figure 2(a). Then we replace the tangles T_i with $T_0 = \infty$ and $T_i = n_i \in \mathbb{Z}$ for $i = 1, \ldots, k$ as in Figure 2(b). We call the resultant diagram a *symmetric union* and write $D_K \cup D_K^*(n_1, \ldots, n_k)$ and the diagram is called a *symmetric union presentation* for K. We say that a knot K is a *symmetric union* if K has a symmetric union presentation.

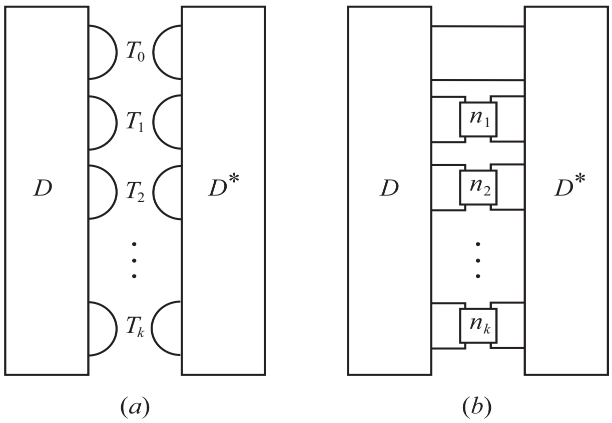

Figure 2: A symmetric union.

3 Banded symmetric unions

Definition 3.1. Let K be a non-trivial knot with a symmetric union presentation $D_K \cup D_K^*(n_1, \cdots, n_k)$. If $D_K \cup D_K^*(\infty, \cdots, \infty)$ is a trivial link diagram, then we call K a *k-banded symmetric union* (or simply a *banded symmetric union*) and $D_K \cup D_K^*(n_1, \cdots, n_k)$ a *k-banded symmetric union presentation.*

Proposition 3.2. *Every symmetric union has a banded symmetric union presentation.*

Proof. Let K be a knot with a symmetric union presentation $D_K \cup D_K^*(n_1, \cdots, n_k)$. Let L be a link with the diagram $D_K \cup D_K^*(\infty, \cdots, \infty)$ obtained from the symmetric union presentation. Then L can be transformed into a trival link by a sequence of *bandings* [3] of K concerning the bands which are symmetric with respect to the symmetry axis on a diagram of symmetric union for K as in Figure 3.

By using an isotopy, K can be transformed into a knot with a banded symmetric union presentation along the bands as in Figure 4.

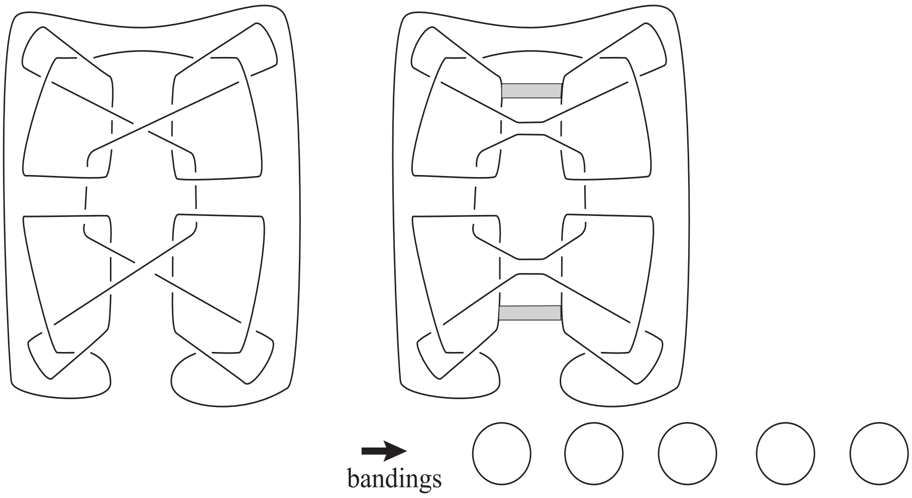

Figure 3: Bandings for a symmetric union.

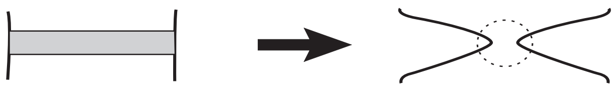

Figure 4: Transforming K along bands.

4 Knots with symmetric ribbon number two.

For an n-component link L, let $\bar{V}_L(t) = V_L(t)/V_{O_n}(t)$ where O_n is an n-component trivial link.

Proof of Theorem 1.1. By [5, Theorem 1.1], we know that

$$V_{\bar{K}}(t) = (1-(-1)^n t^{-n})\bar{V}_{D\cup D^*(\infty)}(t) + (-1)^n t^{-n} V_{D\cup D^*(0)}(t).$$

Since $V_{D\cup D^*(0)} = V_K(t)V_K(t^{-1})$ and $\bar{V}_{D\cup D^*(\infty)}(t) = 1$, we have $V_{\bar{K}}(t) = 1-(-1)^n t^{-n}+(-1)^n t^{-n} V_K(t)V_K(t^{-1})$. Let $a =$*max-deg*$V_{K\sharp K^*}(t)$. Then note that *min-deg*$V_{K\sharp K^*}(t) = -a$ and $a \geq 0$. If $a > 0$, then *red-deg*$V_{\bar{K}}(t) = a-(-a) = 2a$. If $a = 0$, then $V_{\bar{K}}(t) = 1$. Thus we know that if $V_{\bar{K}}(t) \neq 1$, then *red-deg*$V_{\bar{K}}(t) = 2a = 2$*max-deg*$V_{K\sharp K^*}(t)$.

Remark 4.1. Lamm found that symmetric union presentations of all ribbon knots with minimal crossing number equal to or less than ten [4]. It is easily seen that all of them are represented by a symmetric union of an alternating knot, the knots $6_1, 8_8, 8_{20}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{137}$ and 10_{140} have 1-banded symmetric union presentations, the knots $8_9, 9_{27}, 10_{42}, 10_{48}, 10_{99}, 10_{123}, 10_{129}, 10_{155}$ have 2-banded symmetric union presentations, and $9_{41}, 10_{75}$ and 10_{153} have 3-banded symmetric union presentations.

Proof of Theorem 1.2. By the definition of the Jones polynomial, we have $V_{\bar{K}}(t) = \{(-A^3)^{-w(D_K\cup D_K^*(m,n))} < D_K \cup D_K^*(m,n) >\}_{t^{1/2}=A^{-2}}$ and $w(D_K \cup D_K^*(m,n)) = -m-n$. As in the proof of [5, Theorem 1.1], we can calculate $< D_K \cup D_K^*(m,n) >$ as follows.

$< D_K \cup D_K^*(m,n) >= A^m < D_K \cup D_K^*(0,n) > + \dfrac{(-A^{-3})^m - A^m}{-A^{-2}-A^2} < D_K \cup D_K^*(\infty,n) >$,

$< D_K \cup D_K^*(0,n) >= A^n < D_K \cup D_K^*(0,0) > + \dfrac{(-A^{-3})^n - A^n}{-A^{-2}-A^2} < D_K \cup D_K^*(0,\infty) >$ and

$< D_K \cup D_K^*(\infty, n) >= A^n < D_K \cup D_K^*(\infty, 0) > + \dfrac{(-A^{-3})^n - A^n}{-A^{-2} - A^2} < D_K \cup D_K^*(\infty, \infty) >.$
So we have
$(-A^3)^{m+n} < D_K \cup D_K^*(m, n) >= (-1)^{m+n} A^{4(m+n)} < D_K \cup D_K^*(0, 0) > +$
$(-1)^m A^{4m} \dfrac{1 - (-1)^n A^{4n}}{-A^{-2} - A^2} < D_K \cup D_K^*(0, \infty) > + (-1)^n A^{4n} \dfrac{1 - (-1)^m A^{4m}}{-A^{-2} - A^2} < D_K \cup D_K^*(\infty, 0) > +$
$\dfrac{(1 - (-1)^m A^{4m})(1 - (-1)^n A^{4n})}{(A^{-2} + A^2)^2} < D_K \cup D_K^*(\infty, \infty) >.$
Then $V_{\bar{K}}(t) = (-1)^{m+n} t^{-m-n} V_{D_K \cup D_K^*(0,0)}(t) + (-1)^m t^{-m} (1 - (-1)^n t^{-n}) \bar{V}_{D_K \cup D_K^*(0,\infty)}(t) +$
$(-1)^n t^{-n} (1 - (-1)^m t^{-m}) \bar{V}_{D_K \cup D_K^*(\infty,0)}(t) + (1 - (-1)^m t^{-m})(1 - (-1)^n t^{-n}) \bar{V}_{D_K \cup D_K^*(\infty,\infty)}(t).$
By the assumption, we have
$\bar{V}_{D_K \cup D_K^*(0,\infty)}(t) = \bar{V}_{D_K \cup D_K^*(\infty,0)}(t) = \bar{V}_{D_K \cup D_K^*(\infty,\infty)}(t) = 1$ and $V_{D_K \cup D_K^*(0,0)}(t) = V_{K \sharp K^*}(t)$. Thus we have $V_{\bar{K}}(t) = (-1)^{m+n} t^{-m-n} V_{K \sharp K^*}(t) + (-1)^m t^{-m} (1 - (-1)^n t^{-n}) + (-1)^n t^{-n} (1 - (-1)^m t^{-m}) + (1 - (-1)^m t^{-m})(1 - (-1)^n t^{-n}) = 1 - (-1)^{m+n} t^{-m-n} + (-1)^{m+n} t^{-m-n} V_{K \sharp K^*}(t)$. By using the same argument as in the proof of Theorem 1.1, we have *red-deg*$V_{\bar{K}}(t) = 2a = 2$*max-deg*$V_{K \sharp K^*}(t)$ if $V_{\bar{K}}(t) \neq 1$.

Definition 4.2. The *symmetric ribbon number* of a symmetric union K is defined as the minimal number n for any n-banded symmetric union presentation for K.

Remark 4.3. It is easily seen that the *ribbon number* [5] of a symmetric union is equal to or less than the symmetric ribbon number.

Proposition 4.4. *There exists an infinitely many ribbon number one knots with symmetric ribbon number two.*

Proof. We consider the symmetric union K_n which can be represented by $D \cup D^*(n, -1) (n \geq 1)$ as in Figure 5.

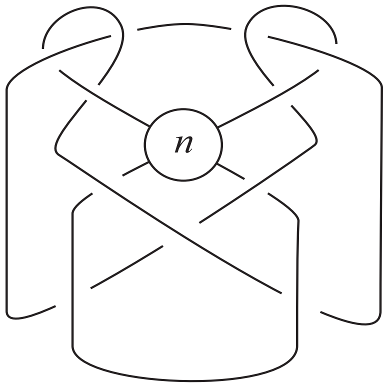

Figure 5: K_n

Since $D \cup D^*(n, \infty)$ and $D \cup D^*(\infty, \infty)$ are trivial links, we know that K_n is a ribbon number one knot and the symmetric ribbon number is equal to or less than two. By the definition of the bracket polynomial and the calculation in the proof of Theorem 1.2, we can calculate as follows.
$< D_{K_n} >= A^n < D_K \cup D_K^*(0, -1) > + \dfrac{(-A^{-3})^n - A^n}{-A^{-2} - A^2} < D_K \cup D_K^*(\infty, -1) >$ where,
$< D \cup D^*(0, -1) >= -A^{19} + 2A^{15} - 4A^{11} + 6A^7 - 8A^3 + 8A^{-1} - 7A^{-5} + 6A^{-9} - 4A^{-13} + 2A^{-17} - A^{-21},$
$< D \cup D^*(\infty, -1)) >= A^{17} - A^{13} + A^9 + A - A^{-7} + A^{-11} - A^{-15} + A^{-19}.$
So we have
$< D_{K_n} >= A^n(-A^{19} + 2A^{15} - 4A^{11} + 6A^7 - 8A^3 + 8A^{-1} - 7A^{-5} + 6A^{-9} - 4A^{-13} + 2A^{-17} - A^{-21}) + ((-A^{-3})^n -$

$A^n)(A^{15} - 2A^{11} + 3A^7 - 3A^3 + 4A^{-1} - 4A^{-5} + 3A^{-9} - 2A^{-13} + A^{-17})$.
Then we note that *red-deg*$< D_{K_n} >= 4n + 36$. So *red-deg*$V_{K_n}(t) = n + 9$. Thus we know that $K_{2m}(m \geq 1)$ does not have a 1-banded symmetric union presentation by Theorem 1.1.

Remark 4.5. For example, 10_{129} and 10_{155} have 2-banded symmetric union presentations. However we do not know if the symmetric ribbon numbers of 10_{129} and 10_{155} are two.

Acknowledgements. The author is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research(C), 2022-2024(22K03310).

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