Examples of Rational Algebraic Varieties

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Abstract

We give some examples of rational algebraic varieties. Example 1 is well known. Examples 2, 3 and 4 below are our examples.

Let K be a field. For any reduced and irreducible scheme M over Spec K the local ring at the generic point of M is a field, for which we write $K(M)$. One calls $K(M)$ the function field of M over K. Let $K[t_1, t_2, ..., t_n]$ denote the polynomial ring of n variables over K. Let $K(t_1, t_2, ..., t_n)$ denote the quotient field of $K[t_1, t_2, ..., t_n]$. It is called usually the rational function field over K of n variables. One calls M a rational algebraic variety of dimension n over K if $K(M)$ is isomorphic to the rational function field $K(t_1, t_2, ..., t_n)$ over K.

Example 1. It is well known that the projective variety $\sum_{j=1}^{n} x_j^2 = x_{n+1}^2$ for $n \geqq 2$ over a field K with char.$K \neq 2$ is rational and irreducible. For we have $\sum_{j=1}^{n-1} x_j^2 + y_n y_{n+1} = 0$ where $y_n = x_n + x_{n+1}$ and $y_{n+1} = x_n - x_{n+1}$.

Example 2. We consider the projective variety $u^2z - 2uxy + x^3 - xz^2 + y^2z = 0$ of dimension 2 over any field K. It is irreducible. We see this easily since $u^2z - 2uxy + x^3 - xz^2 + y^2z$ has degree 2 with respect to u. This algebraic variety is rational.

Let a, b and c be letters algebraically independent over K. Let X, Y, Z and U are letters algebraically independent over K. We define the K algebra homomorphism $f : K[X, Y, Z, U] \rightarrow K[a, b, c]$ by $P(X, Y, Z, U) \mapsto P(a^2b + bc^2, ac^2 - b^2c - a^3, -2abc + b^3, -ab^2 + a^2c - c^3)$. We obtain Ker$f$ =the principal ideal generated by $U^2Z - 2UXY + X^3 - XZ^2 + Y^2Z$, and $\frac{K[Y,Z,U]}{(U^2Z-2UY+1-Z^2+Y^2Z)}$ $\cong K[-\frac{-ac^2+b^2c+a^3}{a^2b+bc^2}, \frac{-2ac+b^2}{a^2+c^2}, -\frac{ab^2-a^2c+c^3}{a^2b+bc^2}] \subset K(\frac{a}{b}, \frac{c}{b})$. Hence the function field of this variety is isomorphic to $K(-\frac{-ac^2+b^2c+a^3}{a^2b+bc^2}, \frac{-2ac+b^2}{a^2+c^2}, -\frac{ab^2-a^2c+c^3}{a^2b+bc^2})$ whose transcendental degree over K is 2. This is a rational function field of degree 2. This variety is a rational algebraic surface.

Example 3. We consider the projective algebraic variety of dimension 3 over any field K defined by $-u^3y + 2u^2vx + u^2z^2 - uv^3 - uvyz - 3ux^2z + 2uxy^2 + 2v^2xz + v^2y^2 - 3vx^2y - vz^3 + x^4 + 2xyz^2 - y^3z = 0$. We write $g(x, y, z, u, v)$ for the left side of this equation. Let X, Y, Z, U and V be letters algebraically independent over K. We have $g(X, Y, Z, U, V) = X^4 + (-3UZ - 3VY)X^2 + (2U^2V + 2UY^2 + 2V^2Z + 2YZ^2)X + (U^2Z^2 - U^3Y - UV^3 - UVYZ + V^2Y^2 - VZ^3 - Y^3Z)$. We see $g(0, Y, Z, 0, V)$ is irreducible. We see $g(0, Y, Z, U, V)$ is irreducible. We obtain $g(X, Y, Z, U, V)$ is irreducible.

This algebraic variety is rational.

Let a, b, c and d be letters algebraically independent over K. Let X, Y, Z, U and V be letters algebraically independent over K. We define the K algebra homomorphism $\varphi : K[X, Y, Z, U, V] \to K[a, b, c, d]$ by $P(X, Y, Z, U, V) \mapsto P(a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd, b^3c + bd^3 - c^2d^2 - a^4 - 2abc^2 - 2ab^2d + 3a^2cd, ad^3 + c^3d - a^2c^2 - b^4 + 3ab^2c - 2a^2bd - 2bcd^2, ab^3 + a^3d - b^2d^2 - c^4 - 2a^2bc - 2acd^2 + 3bc^2d, a^3c + bc^3 - a^2b^2 - d^4 + 3abd^2 - 2ac^2d - 2b^2cd)$. We obtain Ker φ =the principal ideal generated by $g(X, Y, Z, U, V)$, and

$$\frac{K[Y, Z, U, V]}{(g(1, Y, Z, U, V))} \cong K[-\frac{-b^3c - bd^3 + c^2d^2 + a^4 + 2abc^2 + 2ab^2d - 3a^2cd}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd}, -\frac{-ad^3 - c^3d + a^2c^2 + b^4 - 3ab^2c + 2a^2bd + 2bcd^2}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd}, -\frac{-ab^3 - a^3d + b^2d^2 + c^4 + 2a^2bc + 2acd^2 - 3bc^2d}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd}, -\frac{-a^3c - bc^3 + a^2b^2 + d^4 - 3abd^2 + 2ac^2d + 2b^2cd}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd}] \subset K(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}).$$

Hence the function field of this variety is isomorphic to

$$K(-\frac{-b^3c - bd^3 + c^2d^2 + a^4 + 2abc^2 + 2ab^2d - 3a^2cd}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd}, -\frac{-ad^3 - c^3d + a^2c^2 + b^4 - 3ab^2c + 2a^2bd + 2bcd^2}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd}, -\frac{-ab^3 - a^3d + b^2d^2 + c^4 + 2a^2bc + 2acd^2 - 3bc^2d}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd}, -\frac{-a^3c - bc^3 + a^2b^2 + d^4 - 3abd^2 + 2ac^2d + 2b^2cd}{a^3b + ac^3 + b^3d + cd^3 - a^2d^2 - b^2c^2 + abcd})$$

whose transcendental degree over K is 3. It is a subfield of $K(\frac{a}{d},\frac{b}{d},\frac{c}{d})$. Further we obtain the function field of this variety is rational. Hence this algebraic variety is a rational threefold.

Remark 1.

$$\frac{a}{d} = -\frac{ux^2+vz^2+y^3-uvy-2xyz}{uy^2+v^2z+x^3-uxz-2vxy},$$
$$\frac{b}{d} = -\frac{u^2y-uz^2-xy^2+x^2z-uvx+vyz}{uy^2+v^2z+x^3-uxz-2vxy},$$
$$\frac{c}{d} = -\frac{uv^2-u^2x-vy^2+x^2y+uyz-vxz}{uy^2+v^2z+x^3-uxz-2vxy}$$

where $g(x,y,z,u,v)=0$.

Example 4. We consider the projective algebraic variety of dimension 4 over any field K defined by $u^4x-2u^3vw-2u^3yz+u^2v^3+3u^2vy^2+3u^2w^2z-2u^2x^3+u^2z^3-6uv^2xy+6uvwx^2-2uw^3x-6uwxz^2+6ux^2yz-2uxy^3-v^4z+2v^3wy+v^3x^2-3v^2w^2x+3v^2xz^2+vw^4-2vwy^3-4vx^3z+3vx^2y^2-vz^4-2w^3yz+3w^2x^2z+3w^2xy^2-4wx^3y+2wyz^3+x^5+x^2z^3-3xy^2z^2+y^4z=0.$

We write $h(x,y,z,u,v,w)$ for the left side of this equation. Let X, Y, Z, U, V and W be letters algebraically independent over K. We have

$$h(X,Y,Z,U,V,W) = \begin{vmatrix} X & Y & Z & U & V \\ W & X & Y & Z & U \\ V & W & X & Y & Z \\ U & V & W & X & Y \\ Z & U & V & W & X \end{vmatrix}.$$

We see $h(X,Y,Z,0,0,0)$ is irreducible. We obtain $h(X,Y,Z,U,V,W)$ is irreducible.

This algebraic variety is rational.

Let a, b, c, d and e be letters algebraically independent over K. We define the K algebra homomorphism $\psi : K[X,Y,Z,U,V,W] \to K[a,b,c,d,e]$ by

$X \mapsto de^4+a^4b-bd^4-b^4d+b^3c^2+c^2d^3-2abe^3+2ab^3e+2ad^3e-2a^3de-2c^3de-2abc^3+3bc^2e^2+3a^2c^2d=\alpha$

$Y \mapsto -a^5+4ea^3c+2a^3d^2-3ea^2b^2-6a^2bcd-a^2c^3-e^3a^2+2ab^3d+3ab^2c^2+6e^2abd-3e^2ac^2-ad^4-b^4c-3eb^2d^2+2bcd^3+ec^4-c^3d^2+e^4c-e^3d^2=\beta$

$Z \mapsto -2ea^3b-2a^3cd+3a^2b^2d+3a^2bc^2+3e^2a^2d-4ab^3c-6eabd^2+2acd^3+b^5-2e^2b^3+6eb^2cd+b^2d^3-2ebc^3-3bc^2d^2+e^4b+c^4d-2e^3cd+e^2d^3=\gamma$

$U \mapsto ea^4-2a^3bd-a^3c^2+3a^2b^2c-3e^2a^2c-ab^4+2abd^3+4eac^3-3ac^2d^2+e^4a+2eb^3d-3eb^2c^2-3b^2cd^2+4bc^3d-2e^3bd-c^5-e^3c^2+3e^2cd^2-ed^4=\delta$

$V \mapsto a^4d-2a^3bc+a^2b^3+3e^2a^2b-2a^2d^3-6eab^2d+6abcd^2-2ac^3d-2e^3ad+2eb^3c+b^3d^2-3b^2c^2d+bc^4-2e^3bc+3e^2bd^2+3e^2c^2d-4ecd^3+d^5=\varepsilon$

$W \mapsto a^4c - a^3b^2 - e^2a^3 + 6ea^2bd - 3ea^2c^2 - 3ab^2d^2 + ac^4 + 4e^3ac - 3e^2ad^2 - eb^4 + 2b^3cd - b^2c^3 + 2e^3b^2 - 6e^2bcd + 2ebd^3 - e^2c^3 + 3ec^2d^2 - cd^4 - e^5 = \mu$.

We obtain Ker ψ =the principal ideal generated by $h(X,Y,Z,U,V,W)$, and $\frac{K[Y,Z,U,V,W]}{(h(1,Y,Z,U,V,W))} \cong K[\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}, \frac{\delta}{\alpha}, \frac{\epsilon}{\alpha}, \frac{\mu}{\alpha}] \subset K(\frac{a}{e}, \frac{b}{e}, \frac{c}{e}, \frac{d}{e})$. Hence the function field of this variety is isomorphic to $K(\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}, \frac{\delta}{\alpha}, \frac{\epsilon}{\alpha}, \frac{\mu}{\alpha})$ whose transcendental degree over K is 4. It is a subfield of $K(\frac{a}{e}, \frac{b}{e}, \frac{c}{e}, \frac{d}{e})$. Further we obtain the function field of this variety is rational. Hence this is a rational algebraic variety of dimension 4.

Remark 2.

$$\frac{a}{e} = \frac{-u^2vx+u^2w^2+2uvyz-2uwxz-2uwy^2+2ux^2y+v^2xz-v^2y^2-vw^2z+2vwxy-vx^3-vz^3+2wyz^2+x^2z^2-3xy^2z+y^4}{u^2wy-u^2x^2-uv^2y+2uvwx-uw^3-uwz^2+2uxyz-uy^3+v^2z^2-2vwyz-2vx^2z+2vxy^2+2w^2xz+w^2y^2-3wx^2y+x^4},$$

$$\frac{b}{e} = \frac{u^3x-u^2vw-2u^2yz+2uvy^2+uw^2z-ux^3+uz^3+v^2wz-v^2xy-vw^2y+vwx^2-vyz^2-2wxz^2+wy^2z+2x^2yz-xy^3}{u^2wy-u^2x^2-uv^2y+2uvwx-uw^3-uwz^2+2uxyz-uy^3+v^2z^2-2vwyz-2vx^2z+2vxy^2+2w^2xz+w^2y^2-3wx^2y+x^4},$$

$$\frac{c}{e} = \frac{-u^3w+u^2v^2+u^2xz+u^2y^2-4uvxy+uw^2y+uwx^2-uyz^2-v^3z+v^2wy+v^2x^2-vw^2x+vxz^2+vy^2z-wy^3-x^3z+x^2y^2}{u^2wy-u^2x^2-uv^2y+2uvwx-uw^3-uwz^2+2uxyz-uy^3+v^2z^2-2vwyz-2vx^2z+2vxy^2+2w^2xz+w^2y^2-3wx^2y+x^4},$$

$$\frac{d}{e} = \frac{u^2wz-u^2xy-uv^2z+2uvx^2-uw^2x-uxz^2+uy^2z+v^3y-2v^2wx+vw^3+vwz^2-vy^3-2w^2yz+wx^2z+2wxy^2-x^3y}{u^2wy-u^2x^2-uv^2y+2uvwx-uw^3-uwz^2+2uxyz-uy^3+v^2z^2-2vwyz-2vx^2z+2vxy^2+2w^2xz+w^2y^2-3wx^2y+x^4}$$

where $h(x,y,z,u,v,w) = 0$.

Some Computations.

$$\begin{vmatrix} x & y & z & u & v \\ w & x & y & z & u \\ v & w & x & y & z \\ u & v & w & x & y \\ z & u & v & w & x \end{vmatrix} = u^4x - 2u^3vw - 2u^3yz + u^2v^3 + 3u^2vy^2 + 3u^2w^2z - 2u^2x^3 + u^2z^3 - 6uv^2xy + 6uvwx^2 - 2uw^3x - 6uwxz^2 + 6ux^2yz - 2uxy^3 - v^4z + 2v^3wy + v^3x^2 - 3v^2w^2x + 3v^2xz^2 + vw^4 - 2vwy^3 - 4vx^3z + 3vx^2y^2 - vz^4 - 2w^3yz + 3w^2x^2z + 3w^2xy^2 - 4wx^3y + 2wyz^3 + x^5 + x^2z^3 - 3xy^2z^2 + y^4z,$$

$$\begin{vmatrix} x & y & z & u \\ w & x & y & z \\ v & w & x & y \\ u & v & w & x \end{vmatrix} = u^2wy - u^2x^2 - uv^2y + 2uvwx - uw^3 - uwz^2 + 2uxyz - uy^3 + v^2z^2 - 2vwyz - 2vx^2z + 2vxy^2 + 2w^2xz + w^2y^2 - 3wx^2y + x^4,$$

$$\begin{vmatrix} -v & y & z & u \\ -u & x & y & z \\ -z & w & x & y \\ -y & v & w & x \end{vmatrix} = -u^2vx+u^2w^2+2uvyz-2uwxz-2uwy^2+2ux^2y+v^2xz-$$
$$v^2y^2 - vw^2z + 2vwxy - vx^3 - vz^3 + 2wyz^2 + x^2z^2 - 3xy^2z + y^4,$$

$$\begin{vmatrix} x & -v & z & u \\ w & -u & y & z \\ v & -z & x & y \\ u & -y & w & x \end{vmatrix} = u^3x-u^2vw-2u^2yz+2uvy^2+uw^2z-ux^3+uz^3+v^2wz-$$
$$v^2xy - vw^2y + vwx^2 - vyz^2 - 2wxz^2 + wy^2z + 2x^2yz - xy^3,$$

$$\begin{vmatrix} x & y & -v & u \\ w & x & -u & z \\ v & w & -z & y \\ u & v & -y & x \end{vmatrix} = -u^3w + u^2v^2 + u^2xz + u^2y^2 - 4uvxy + uw^2y + uwx^2 -$$
$$uyz^2 - v^3z + v^2wy + v^2x^2 - vw^2x + vxz^2 + vy^2z - wy^3 - x^3z + x^2y^2,$$

$$\begin{vmatrix} x & y & z & -v \\ w & x & y & -u \\ v & w & x & -z \\ u & v & w & -y \end{vmatrix} = u^2wz - u^2xy - uv^2z + 2uvx^2 - uw^2x - uxz^2 + uy^2z +$$
$$v^3y - 2v^2wx + vw^3 + vwz^2 - vy^3 - 2w^2yz + wx^2z + 2wxy^2 - x^3y,$$

$$\begin{vmatrix} b & c & d & e & 0 \\ c & d & e & 0 & a \\ d & e & 0 & a & b \\ e & 0 & a & b & c \\ 0 & a & b & c & d \end{vmatrix} = a^4b - 2ea^3d + 3a^2c^2d + 2eab^3 - 2abc^3 - 2e^3ab + 2ead^3 -$$
$$b^4d + b^3c^2 + 3e^2bc^2 - bd^4 - 2ec^3d + c^2d^3 + e^4d,$$

$$\begin{vmatrix} -a & c & d & e & 0 \\ -b & d & e & 0 & a \\ -c & e & 0 & a & b \\ -d & 0 & a & b & c \\ -e & a & b & c & d \end{vmatrix} = -a^5 + 4ea^3c + 2a^3d^2 - 3ea^2b^2 - 6a^2bcd - a^2c^3 - e^3a^2 +$$
$$2ab^3d + 3ab^2c^2 + 6e^2abd - 3e^2ac^2 - ad^4 - b^4c - 3eb^2d^2 + 2bcd^3 + ec^4 - c^3d^2 +$$
$$e^4c - e^3d^2,$$

$$\begin{vmatrix} b & -a & d & e & 0 \\ c & -b & e & 0 & a \\ d & -c & 0 & a & b \\ e & -d & a & b & c \\ 0 & -e & b & c & d \end{vmatrix} = -2ea^3b - 2a^3cd + 3a^2b^2d + 3a^2bc^2 + 3e^2a^2d - 4ab^3c -$$

$6eabd^2+2acd^3+b^5-2e^2b^3+6eb^2cd+b^2d^3-2ebc^3-3bc^2d^2+e^4b+c^4d-2e^3cd+e^2d^3$,

$$\begin{vmatrix} b & c & -a & e & 0 \\ c & d & -b & 0 & a \\ d & e & -c & a & b \\ e & 0 & -d & b & c \\ 0 & a & -e & c & d \end{vmatrix} = ea^4-2a^3bd-a^3c^2+3a^2b^2c-3e^2a^2c-ab^4+2abd^3+4eac^3-3ac^2d^2+e^4a+2eb^3d-3eb^2c^2-3b^2cd^2+4bc^3d-2e^3bd-c^5-e^3c^2+3e^2cd^2-ed^4,$$

$$\begin{vmatrix} b & c & d & -a & 0 \\ c & d & e & -b & a \\ d & e & 0 & -c & b \\ e & 0 & a & -d & c \\ 0 & a & b & -e & d \end{vmatrix} = a^4d-2a^3bc+a^2b^3+3e^2a^2b-2a^2d^3-6eab^2d+6abcd^2-2ac^3d-2e^3ad+2eb^3c+b^3d^2-3b^2c^2d+bc^4-2e^3bc+3e^2bd^2+3e^2c^2d-4ecd^3+d^5,$$

$$\begin{vmatrix} b & c & d & e & -a \\ c & d & e & 0 & -b \\ d & e & 0 & a & -c \\ e & 0 & a & b & -d \\ 0 & a & b & c & -e \end{vmatrix} = a^4c-a^3b^2-e^2a^3+6ea^2bd-3ea^2c^2-3ab^2d^2+ac^4+4e^3ac-3e^2ad^2-eb^4+2b^3cd-b^2c^3+2e^3b^2-6e^2bcd+2ebd^3-e^2c^3+3ec^2d^2-cd^4-e^5.$$

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