ON THE JONES POLYNOMIAL OF SYMMETRIC UNIONS WITH TWO COMPONENTS

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Abstract. A symmetric union is a link with a diagram, obtained from diagrams of a knot in the 3-space and its mirror image. In this paper, we give certain formulas of the Jones polynomial of a link with a symmetric union presentation and consider an invariant of symmetric union, which is called the minimal twisting number and show that there exists a link with a symmetric union presentation such that the minimal twisting number is strictly larger than the sum of the minimal twisting numbers of its components.

Key words: symmetric union, Jones polynomial, ribbon link.

1. Introduction

A symmetric union, which is a generalized operation of the connected sum of a knot in the 3-space and its mirror image, was first introduced by Kinoshita and Terasaka [5]. They showed that the Alexander polynomial depends only on the parity of the number of half-twists of a trivial tangle on the symmetry axis. Recently, Lamm [6] generalized their definition and considered the relationship between a symmetric union and a ribbon link. (See [3] for the definition.) Every link with a symmetric union presentation is a ribbon link and Lamm showed that every ribbon knot with minimal crossing number \( \leq 10 \) has a symmetric union presentation [6][2] and it is known that all two-bridge ribbon knots can be represented as symmetric unions. [7][9].

Let \( \overline{V}_L(t) = V_L(t)/V_O^n(t) \) for an oriented link \( L \) of \( n \) components, where \( V_L(t) \) and \( V_O^n(t) \) are the Jones polynomial of \( L \) and the \( n \)-component trivial link \( O^n \) respectively. (See Section 2 for the definition.)

It is known that the Alexander polynomial of a symmetric union of \( n \) components \( (n \geq 2) \) is zero [6]. In this paper we study the Jones polynomial of links with symmetric union presentations and its topological properties.

Theorem 1.1. Let \( \overline{L} \) be a link with a symmetric union presentation of the form \( D \cup D^*(\infty_2, m) \). Then

\[
\overline{V}_L(t) = (-1)^m t^{-m} \overline{V}_{D \cup D^*(\infty_2, 0)}(t) + (1 - (-1)^m t^{-m}) \overline{V}_{D \cup D^*(\infty_2, \infty)}(t).
\]

A link is called amphicheiral, if it is isotopic to its mirror image. By Theorem 1.1, we have the following.

Theorem 1.2. Let \( \overline{L} \) be a link with a symmetric union presentation of the form \( D \cup D^*(\infty_2, m) \). Then \( t^m \overline{V}_L(t) + (-1)^m \overline{V}_L(t^{-1}) = (t^m + (-1)^m) \overline{V}_{D \cup D^*(\infty_2, 0)}(t) \). In particular, if \( \overline{K} \) is amphicheiral, then \( \overline{V}_\pi(t) = \overline{V}_{D \cup D^*(\infty_2, 0)}(t) \).

By using the similar proof of Theorem 1.2, we can show that following theorem.

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Theorem 1.3. Let $\Gamma$ be a link with a symmetric union presentation of the form $D \cup D^* (\infty_2, m)$. Then $t^m \overline{V}(t) - t^{-m} \overline{V}(t^{-1}) = (t^m - t^{-m}) \overline{V}_{D \cup D^* (\infty_2, \infty)}(t)$. In particular, if $\Gamma$ is amphicheiral, then $\overline{V}(t) = \overline{V}_{D \cup D^* (\infty_2, \infty)}(t)$.

Now we restrict to the special values of the Jones polynomial. We denote a (Laurent) polynomial $f(t)$ evaluated at $r$ by $[f(t)]_{t=r}$.

Theorem 1.4. Let $\Gamma$ be a link with a symmetric union presentation of the form $D \cup D^* (\infty_2, m)$. Then

$$\left[ \frac{d}{dt} \overline{V}(t) \right]_{t=-1} = m \{ \overline{V}_{D \cup D^* (\infty_2, 0)}(-1) - \overline{V}_{D \cup D^* (\infty_2, \infty)}(-1) \}.$$

Corollary 1.5. Let $\Gamma$ be a link with a symmetric union presentation of the form $D \cup D^* (\infty_2, m)$. Then $\left[ \frac{d}{dt} \overline{V}(t) \right]_{t=-1} \equiv 0 \mod 8|\mu|$.

Remark 1.6. These results can be generalized to the case of $D \cup D^* (\infty_\mu, m) \ (\mu \geq 2)$.

In this paper, all knots and links are oriented unless otherwise stated. In Section 2, we give the definitions of the Jones polynomial and a symmetric union. In Section 3, we shall prove Theorem 1.1 and Theorem 1.2. In Section 4, we shall prove Theorem 1.4 and Corollary 1.5. In Section 5, we introduce the minimal twisting number of a link with a symmetric union presentation. It is the smallest number of trivial tangles (with twists) appearing on the axis of a symmetric union presentation of a link, the minimum taken over all symmetric union presentations for the link. We shall show that there exists a symmetric union such that the minimal twisting number is strictly larger than the sum of the minimal twisting numbers of its components.

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2. Definitions

Definition 2.1. Let $L$ be a link in the 3-space. We denote a diagram of $L$ by $D_L$. The bracket polynomial of a diagram of a link, $<D_L>$, can be defined as a polynomial which satisfies the following identities.

i) $<\bigcirc> = 1$,

ii) $<D_L \cup \bigcirc > = -(A^2 + A^{-2}) <D_L>$,

iii) $<\bigtriangleup > = A <\bigtriangledown > + A^{-1} <\bigtriangledown >$.

We define $V_{D_L}(t) \in \mathbb{Z}[t^{1/2},t^{-1/2}]$ by $V_{D_L}(t) = \{ (-A^3)^{-\omega(D_L)} < D_L > \}_t^{1/2=\omega-2}$ for any diagram $D_L$ for $L$, where $\omega$ is the writhe of the diagram. (The writhe is the number of positive crossings of $D_L$ minus the number of negative crossings of $D_L$.) It is shown that $V_{D_L}(t)$ is an invariant of the link [8][4]. Then we denote $V_{D_L}(t)$ by $V_L(t)$ and call it the Jones polynomial of $L$.

Here we define a symmetric union in [6] as follows. We denote the (trivial) tangles made of half twists by integers $n \in \mathbb{Z}$ and the horizontal trivial tangle by $\infty$ as in Figure 1.

Definition 2.2. Let $D$ be an unoriented link diagram and $D^*$ the diagram $D$ reflected at an axis in the plane. If in the symmetric placement of $D$ and $D^*$ we replace the tangles $T_i = 0, (i = 1, \ldots, k)$ on the symmetry axis by $T_i = \infty$ for $i = 1, \ldots, \mu$ and $T_i = m_i \in \mathbb{Z}$ for $i = \mu + 1, \ldots, k$. We call the
result a symmetric union of $D$ and $D^*$ and denote it by $D \cup D^*(\infty_\mu, m_{\mu+1}, \ldots, m_k)$. See Figure 1 in the case when $\mu = 1$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

If a link $L$ has a diagram $D \cup D^*(\infty_\mu, m_{\mu+1}, \ldots, m_k)$, then the diagram is called a symmetric union presentation for $L$ and we say that the link $L$ is a symmetric union.

3. The Jones Polynomial

Proof of Theorem 1.1. By using a skein relation of Kauffman bracket polynomial, we have

\[ < D \cup D^*(\infty_2, m) > = A^{m/|m|} D \cup D^*(\infty_2, m-1) + F_1 < D \cup D^*(\infty_2, \infty) > \]

\[ = (A^{m/|m|})^{|m|} < D \cup D^*(\infty_2, 0) > + (F_1 + \cdots + F_{|m|}) < D \cup D^*(\infty_2, \infty) > \]

\[ = A^m < D \cup D^*(\infty_2, 0) > + (\sum_{j=1}^{|m|} F_j) < D \cup D^*(\infty_2, \infty) > \]

Now we calculate a formula of $\sum_{j=1}^{|m|} F_j$ by considering the unknot instead of $K$ as follows. We assume that $D$ is a diagram as in Figure 2 so that we have a symmetric union of the unknot. We denote the diagram by $D_0$.

\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}

Then the resultant symmetric union is a diagram of the unknot with $r$ crossings where $r = |m|$ such that it can be transformed into a diagram of the unknot with no crossings by $r$ type I Reidemeister
moves. Thus we have

\[< D_0 \cup D_0^*(\infty_2, m) > = (-A^{-3m/|m|}|m|(-A^{-2} - A^2) = (-1)^{|m|}A^{-3m}(-A^{-2} - A^2), \]

\[< D_0 \cup D_0^*(\infty_2, 0) > = -A^{-2} - A^2, \]

\[< D_2 \cup D_0^*(\infty_2, \infty) > = (A^{-2} + A^2)^2. \]

Then we have

\[\sum_{j=1}^{|m|} F_j = \frac{< D_0 \cup D_0^*(\infty_2, m) > - A^m < D_0 \cup D_0^*(\infty_2, 0) >}{< D_0 \cup D_0^*(\infty_2, \infty) >} = \frac{(-1)^{|m|}A^{-3m} - A^m}{-A^{-2} - A^2} = \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2}. \]

Since \(\omega(D \cup D^*(\infty_2, m)) = -m\), we obtain that

\[V_{D \cup D^*(\infty_2, m)}(A) = (-A^3)^m < D \cup D^*(\infty_2, m) > \]

\[= (-A^3)^m \{A^m < D \cup D^*(\infty_2, 0) > + \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2} < D \cup D^*(\infty_2, \infty) > \} \]

\[= (-1)^m A^{4m} < D \cup D^*(\infty_2, 0) > + \frac{1 - (-1)^m A^{4m}}{-A^{-2} - A^2} < D \cup D^*(\infty_2, \infty) > \]

Using \(t^{1/2} = A^{-2}\), we have

\[\nabla_{D \cup D^*(\infty_2, m)}(t) = (-1)^m t^{-m} \nabla_{D \cup D^*(\infty_2, 0)}(t) + (1 - (-1)^m t^{-m}) \nabla_{D \cup D^*(\infty_2, \infty)}(t). \]

**Proof of Theorem 1.2.** The first part of the theorem is obtained as follows. By Theorem 1.1, we have

\[t^m \nabla_K(t) + (-1)^m \nabla_K(t^{-1}) = t^m((-1)^m t^{-m} \nabla_{D \cup D^*(\infty_2, 0)}(t) + (1 - (-1)^m t^{-m}) \nabla_{D \cup D^*(\infty_2, \infty)}(t)) + (-1)^m((-1)^m t^m V_{D \cup D^*(\infty_2, 0)}(t^{-1}) + (1 - (-1)^m t^m) \nabla_{D \cup D^*(\infty_2, \infty)}(t^{-1}))) \]

\[= (-1)^m V_{D \cup D^*(\infty_2, 0)}(t) + (t^m - (-1)^m) \nabla_{D \cup D^*(\infty_2, \infty)}(t) + t^m V_{D \cup D^*(\infty_2, 0)}(t) + ((-1)^m - t^m) \nabla_{D \cup D^*(\infty_2, \infty)}(t)) = (t^m + (-1)^m) V_{D \cup D^*(\infty_2, 0)}(t) \]

The latter part of the theorem follow immediately from the first part because \(V_K(t) = V_K(t^{-1})\) if \(K\) is amphicheiral ([8], p.29).

**4. Evaluation of the derivative at -1**

**Proof of Theorem 1.4.** By Theorem 1.2, we know that

\[\frac{d}{dt} \nabla_K(t) = \frac{d}{dt}((-1)^m t^{-m} \nabla_{D \cup D^*(\infty_2, 0)}(t)) + \frac{d}{dt}((1 - (-1)^m t^{-m}) \nabla_{D \cup D^*(\infty_2, \infty)}(t)). \]
Proof of Corollary 1.5. We call the number Definition 5.1.

Remark 5.3. Example 5.4.

Therefore we have

Theorem 4.1. [1] If \( K \) be a ribbon link, then \( \nabla_K(-1) \equiv 1 \mod 8 \).

Proof of Corollary 1.5. By Theorem 1.4, we have

By Theorem 4.1, we know that \( V_{D_{L \cup \bar{D}^*}}(0) \) and \( \nabla_{D_{L \cup \bar{D}^*}}(0) \equiv 1 \mod 8 \). Thus we have

5. The minimal twisting number

In this section, we introduce the minimal twisting number for a knot with a symmetric union presentation.

Definition 5.1. We call the number \( k − \mu \) of \( D_{K \cup \bar{D}^*}(\infty, m_{\mu+1}, \ldots, m_k) \) the twisting number of the symmetric union. The minimal twisting number of a link \( L \) with a symmetric union presentation is the smallest number of the twisting numbers of all symmetric union presentations for \( L \). We denote it by \( \text{tw}(L) \).

By the definition, we have the following,

Proposition 5.2. The minimal twisting number is an invariant of a symmetric union.

Remark 5.3. Let \( L \) be a link with a symmetric union presentation. If \( \text{tw}(L) = 0 \), then each component of \( L \) is a connected sum of a knot and its mirror image, could possibly be the unknot.

Example 5.4. For each knot \( K \) in \( \{6_1, 8_8, 8_{20}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{137}, 10_{140}, 10_{153}\} \), we have \( \text{tw}(K) = 1 \). (See [6].)

By definition, we can easily see the following.

Proposition 5.5. Let \( \bar{L} \) be a symmetric union with the two components \( K_1 \) and \( K_2 \). then \( K_1 \) and \( K_2 \) are symmetric unions and satisfies \( \text{tw}(K_1) + \text{tw}(K_2) \leq \text{tw}(\bar{L}) \).
Now we consider the following problem.

**Problem.** Does the equality of the inequality of Proposition 5.5 always hold?

**Example 5.6.** Let \( L_m (m \in \mathbb{Z}, m \neq 0) \) be the symmetric union with two components \( K_1 \) and \( K_2 \) such that \( \text{tw}(K_1) = \text{tw}(K_2) = 0 \) as shown in Figure 3. We know that \( \text{tw}(L_m) = 1 \). In fact, by Theorem 1.1, we have

\[
\overline{V}_{L_m}(t) = (-1)^m t^{-m}(3 - t^{-3} + t^{-2} - t^{-1} - t + t^2 - t^3) + (1 - (-1)^m t^{-m})(13 - t^{-5} + 3t^{-4} - 6t^{-3} + 9t^{-2} - 11t^{-1} - 11t + 9t^2 - 6t^3 + 3t^4 - t^5).
\]

If \( m > 0 \), then the maximal degree is 5 and the minimal degree is \(-m - 5\). On the one hand, if \( m < 0 \), then the maximal degree is \(5 - m\) and the minimal degree is \(-5\). In particular, we know that \( L_m \) is not amphicheiral. Thus we have \( \text{tw}(L_m) = 1 \).

\[ L_m \]

**Figure 3**

Now we consider the following symmetric union \( \hat{L} \) with two components \( K_1 \) and \( K_2 \) such that \( \text{tw}(K_1) = 1 \) and \( \text{tw}(K_2) = 0 \). We know that \( \text{tw}(\hat{L}) \geq 1 \). We can show that \( \hat{L} \) cannot have a presentation \( D_K \cup D_K^{*}(\infty_2, m) \) if \( |m| \neq 1 \) by Theorem 1.2 and Corollary 1.5. However we do not know if \( \hat{L} \) has a presentation \( D_K \cup D_K^{*}(\infty_2, \pm 1) \) at this moment.
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