Signed Gordian distances, the Jones polynomial and Rasmussen invariant of knots

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ABSTRACT

We define signed Gordian distances and give criteria for the Gordian distance for knots by using the Jones polynomial, the Q-polynomial and the Rasmussen invariant of a knot. As a result, we have new values of the Gordian distance with small crossing number.

Keywords: Knot; Gordian distance; Jones polynomial

1. Introduction

A link is a closed oriented 1-manifold embedded smoothly in the 3-sphere $S^3$. A knot is a link with one connected component. A diagram of a link is a generic projection of a link to the 2-sphere in $S^3$ with signed double points, called positive (or negative) crossings as in Figure 1. Let $K$ and $K'$ be two knots in $S^3$. The Gordian distance from $K$ to $K'$, denoted by $d_G(K, K')$, is the minimum number of crossing changes needed to transform a diagram of $K$ into that of $K'$, where the minimum is taken over all diagrams of $K$ and $K'$. A positive crossing change (or negative crossing change) of a crossing is changing a positive (or a negative) crossing of a diagram to a negative (or a positive) crossing. We define $d_{+/-}(K, K')$ (or $d_{-/+}(K, K')$) as the minimum number of positive (or negative) crossing changes needed to transform a diagram of $K$ into that of $K'$ by $d_G(K, K')$ crossing changes, where the minimum is taken over all diagrams of $K$ and $K'$. (See [15] in the case when $K'$ is a trivial knot.)

The Jones polynomial $V$ is a Laurent polynomial in one variable $t$ of a link can be defined by the following relation.

1. $V(\bigcirc; t) = 1$;
2. $t^{-1}V(L_+; t) - tV(L_-; t) = -(t^{-1/2} - t^{1/2})V(L_0; t)$.

Here $L_+$, $L_-$ and $L_0$ are three links with diagrams differing only near a crossing as in Figure 2.

The Jones polynomial can be calculated from the Kauffman bracket $\langle \rangle$ [6]. Let $D$ be an unoriented diagram of a link. Then the Laurent polynomial in one variable $A$ is defined by the following relation:

1. $\langle \bigcirc \rangle = 1$,
2. $\langle \bigcirc \cup D \rangle = -(A^2 + A^{-2})\langle D \rangle$,
3. $\langle \bigotimes \rangle = A\langle \bigcirc \rangle \langle \bigcirc \rangle + A^{-1}\langle \bigotimes \rangle$. 

Figure 1

positive crossing  negative crossing
Then the Jones polynomial is obtained as follows:

\[ V(L; t) = (-t^{-3/4})^{-w(D)} < D > |_{A=t^{-1/4}}, \text{ where } w(D) \text{ is the writhe of } D. \]

Let \( \omega = e^{\pi \sqrt{-1}/3} \) and \( \delta = \frac{\sqrt{5} - 1}{2} \). We need the following theorem.

**Theorem 1.1.** ([7]) For any link \( K \), we have \( V(K; \omega) = \pm (\sqrt{-3})^d \cdot c(K)^{-1} \) for some non-negative integer \( d \), where \( c(K) \) denotes the number of components of \( K \).

In this paper, we show the following.

**Theorem 1.2.** Let \( K \) and \( K' \) be two knots in \( S^3 \). Suppose that \( K' \) is obtained from \( K \) by a positive crossing change.

Let \( f(t) = \frac{tV(K'; t) - V(K; t)}{t - 1} \). Then \( f(\omega) = \pm \omega^{f(1)}(\sqrt{-3})^d \) for some non-negative integer \( d \).

Let \( K \) and \( K' \) be two knots in \( S^3 \). By Theorem 1.1, we may assume that \( V(K; \omega) = (-1)^{t_1}(\sqrt{-3})^{d_1} \) and \( V(K'; \omega) = (-1)^{t_2}(\sqrt{-3})^{d_2} \) for some non-negative integers \( t_1, t_2, d_1 \) and \( d_2 \). Then we have the following theorem which generalizes a result of Traczyk [15].

**Theorem 1.3.** If \( d_G(K, K') = d_1 - d_2 \geq 1 \), then \( d_{-1}(K, K') \equiv t_1 - t_2 \mod 2 \).

The following theorem is a generalization of a result of Stoimenow [13].

**Theorem 1.4.** Let \( K \) and \( K' \) be knots in \( S^3 \). If \( Q(K; \delta)/Q(K'; \delta) = -(\sqrt{5})^k \), then \( d_G(K, K') > |k| \).

Two links are **concordant** if there is a smooth embedding

\[ (nS^1) \times [0, 1] \to S^3 \times [0, 1] \]

which restricts to the given links

\[ (nS^1) \times \{i\} \to S^3 \times \{i\} \]

where \( i = 0, 1 \). The set of concordance classes of knots forms an abelian group under connected sum. The group is called the knot concordance group.

Rasmussen has defined a concordance invariant \( s(K) \) of a knot \( K \) from Lee’s cohomology [10]. We call the invariant the **Rasmussen invariant**. Main properties of Rasmussen invariant are summarized as follows.
Theorem 1.5. Let $K$, $K_1$, and $K_2$ be three knots in $S^3$. Then we have the following.

1. The Rasmussen invariant $s$ induces a homomorphism from the knot concordance group to $\mathbb{Z}$;
2. $|s(K)| \leq 2g_4(K)$, where $g_4(K)$ is the slice genus of $K$;
3. If $K$ is alternating, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical knot signature of $K$;
4. If $K_2$ is obtained from $K_1$ by performing a single positive crossing change, then $s(K_1) - s(K_2) \in \{0, 2\}$.

See [9] for $\sigma(K)$. We have the following by making use of this theorem.

Theorem 1.6. Let $K$ and $K'$ be two knots in $S^3$. Let $s(K, K') = \frac{s(K) - s(K')}{2}$. Then $d_{+-}(K, K') \geq s(K, K')$ and $d_{--}(K, K') \geq -s(K, K')$. In particular, $d_{2-}(K, K') \geq |s(K, K')|$.

2. Proofs

Proof of Theorem 1.2. By a hypothesis, we may consider that a diagram $D'$ of $K'$ is obtained from a diagram $D$ of $K$ by a single positive crossing change. We may assume $D$ to have zero writhe adding kinks if necessarily.

Note that $V(K; A^{-3}) = <D>$ and $A^{-6}V(K; A^{-4}) = <D'>$ since $w(D') + 2 = w(D) = 0$. By using the Kauffman bracket relation, we have

\begin{align*}
(1) & A\left(\begin{array}{c}
\varepsilon
\end{array}\right) + A^{-1}\left(\begin{array}{c}
\varepsilon
\end{array}\right) = V(K; A^{-4}), \\
(2) & A^{-1}\left(\begin{array}{c}
\varepsilon
\end{array}\right) + A\left(\begin{array}{c}
\varepsilon
\end{array}\right) = A^{-6}V(K'; A^{-4}).
\end{align*}

Thus $(A^2 - A^{-2})\left(\begin{array}{c}
\varepsilon
\end{array}\right) = A^{-5}V(K'; A^{-4}) - A^{-1}V(K'; A^{-4}).$

Then we obtain $\left(\begin{array}{c}
\varepsilon
\end{array}\right) = \frac{A^{-3}[A^{-2}V(K'; A^{-4}) - A^{-2}V(K; A^{-4})]}{A^2 - A^{-2} - 1} = -A^{-3}\left(\frac{tV(K'; t) - V(K; t)}{t - 1}\right)$. Let $f(t) = tV(K'; t) - V(K; t)$. Note that $f(t) = \frac{t(V(K'; t) - 1)}{t - 1} - V(K; t) - 1 + 1$. By a result in ([5], §12), $V(K; t) - 1, V(K'; t) - 1$ and $V(\tilde{K}; t) - 1$ have $(t - 1)(t^3 - 1)$ as factors. Then we know that there exists a knot $\tilde{K}$ with the diagram $\left(\begin{array}{c}
\varepsilon
\end{array}\right)$ such that $V(\tilde{K}; t) = t^n f(t)$ for some integer $n$. Here we have $V'(\tilde{K}; t) = n(t^{n-1}f(t) + t^n f'(t))$. By substituting 1, we have $V'(\tilde{K}; 1) = nf(1) + f'(1)$. Thus $f(1) = 1$ and $V'(\tilde{K}; 1) = 0$. Therefore $n = -f'(1)$, and hence we have $V(\tilde{K}; t) = t^{f'(1)} f(t)$. By Theorem 1.1, we know that $V(\tilde{K}; \omega) = \omega^{-f'(1)} f(\omega) = \pm(\sqrt{-3})^d$ for some non-negative integer $d$.

Proof of Theorem 1.3. By an assumption that $d_G(K; K') = d_1 - d_2$, we have a sequence of $d_1 - d_2 + 1$ knots $K = K_{d_1-d_2}, K_{d_1-d_2-1}, \ldots, K_1, K_0 = K'$ such that $K = K_{d_1-d_2}$, $K_0 = K'$ and $d_G(K_j, K_{j-1}) = 1$ ($1 \leq j \leq d_1 - d_2$). By Theorem 1.1, we may assume that $V(K_1; \omega) = (-1)^{s_1}(\sqrt{-3})^{\ell_1}$ for some integers $s_1$ and $\ell_1$ ($0 \leq i \leq d_1 - d_2$), where $\ell_{d_1-d_2} = d_1$, $\ell_0 = d_2$, $s_{d_1-d_2} = t_1$ and $s_0 = t_2$. First we show that $\ell_j - \ell_{j-1} = 1$ ($1 \leq j \leq d_1 - d_2$) as follows.

Let $\ell_j - \ell_{j-1} = n_j$ ($1 \leq j \leq d_1 - d_2$) and suppose that $n_j \geq 2$ for some $j$. In the case when $K_{j-1}$ is obtained from $K_j$ by a positive crossing change, by substituting $\omega$ for $t$ in the second relation of the definition of the Jones polynomial, we have
Proof of Theorem 1.4. We show the theorem by an induction with respect to the Gordian distance. Suppose that $d_G(K, K') = 1$. By an argument in the proof of Theorem 4.1 [13], $Q(K, \delta)/Q(K', \delta) \in \{ \pm 1, -(-\sqrt{3})^{\pm 1} \}$ if $K'$ is obtained from $K$ by a single crossing change. Thus if $Q(K, \delta)/Q(K', \delta) = -(-\sqrt{3})^k$, then $|k| = 0$. Therefore $d_G(K, K') = 1 > 0 = |k|$. Now we assume that the result holds in the case when $d_G(K, K') = m - 1$ for a positive integer $m \geq 2$. Suppose that $d_G(K, K') = m \geq 1$ and $Q(K, \delta)/Q(K', \delta) = -(-\sqrt{3})^k$ for some
integer $k'$. Then there exist $m + 1$ knots $K_0, K_1, \ldots, K_m$ such that $K = K_0$, $K' = K_m$ and $d_G(K_i, K_{i+1}) = 1$ ($0 \leq i < m$). Note that $Q(K, \delta)/Q(K', \delta) = \frac{Q(K_0, \delta)}{Q(K_0, \delta) \times Q(K_{m-1}, \delta)}$ for some integer $\delta$, then we have $|\delta| < m - 1$. We also detect $d_G(K_0, K_{m-1}) = m - 1$ and $d_G(K_{m-1}, K_m) = 1$. Then we have $|k' - \delta| \leq 1$, and hence $|k'| \leq |k' - \delta| + |\delta| < m = d_G(K_0, K_m)$. If $Q(K_0, \delta)/Q(K_{m-1}, \delta) \neq (-\sqrt{5})^\delta$ for any integer $\delta$, then we have $Q(K_0, \delta)/Q(K_{m-1}, \delta) = (-\sqrt{5})^\delta$ for some integer $\delta$, thus $d_G(K_0, K_{m-1}) = |\delta| + 1$. Then we have $|k'| \leq |k' - \delta| + |\delta| \leq m - 1$. Thus we have $k' - \delta = 0$. Therefore $|k'| \leq |k' - \delta| + |\delta| \leq d_G(K_0, K_{m-1}) = m - 1 < m = d_G(K_0, K_m)$. This completes the proof.

**Proof of Theorem 1.6.** If $s(K, K') \geq 0$, then we need to perform at least $s(K, K')$ positive crossing changes to obtain $K'$ from $K$ by Theorem 1.5(4). Thus we have $d_{+}(K, K') \geq s(K, K')$. If $s(K, K') < 0$, then, by using the same idea, we have $d_{-}(K', K') = d_{-}(K', K) \geq s(K', K) = -s(K, K')$.

### 3. Examples

For a knot $K$, $K^*$ denotes the mirror image of $K$. In this section, we detect $d_G(3_1^*4_1, 5_1), d_G(3_1^*3_1, 4_1), d_G(3_1^*3_1, 4_1)$ and $d_G(3_1^*3_1, 7_1)$ which are unknown values in a table of Darcy and Sumners in [3] by using Theorems 1.2, 1.3, 1.4 and 1.6. We also detect $d_G(4_1^*4_1, 3_1), d_G(4_1^*4_1, 3_1^*), d_G(4_1^*4_1, 5_2), d_G(4_1^*4_1, 5_2^*), d_G(4_1^*4_1, 6_1)$ which are unknown values in a table [2] of Darcy by using Theorem 1.4. Moreover we determine some other Gordian distances, by using Theorem 1.6, which cannot be obtained from Theorems 1.2, 1.3 and 1.4. (See Figures 3, 4, 5, 6 and 7 for a diagram of a knot $A_{ab}$.)

**Example 3.1.**

1. $d_G(3_1^*4_1, 5_1) = 2$.
2. $d_{+}(3_1^*4_1, 5_1) = 0$ and $d_{-}(3_1^*4_1, 5_1) = 1$.

We prove (1) by using Theorems 1.2 and 1.5 as follows. It is easily seen that $d_G(3_1^*4_1, 5_1) \leq 2$ since $d_G(0_1, 4_1) = 1$ and $d_G(3_1, 5_1) = 1$. Suppose that $3_1^*4_1$ is obtained from $5_1$ by a positive crossing change. Then $\frac{\omega V(3_1^*4_1; \omega) - V(5_1; \omega)}{\omega - 1} = \frac{\omega(-3\sqrt{3}) - (-1)}{\omega - 1} = \frac{-1 - \omega(-3\sqrt{3})}{1 - \omega} = \frac{-1 + \sqrt{3}\omega}{\sqrt{3} - \sqrt{3}} = \omega(\sqrt{3}\omega - 1) = -2 - \sqrt{3}.$

This contradicts to Theorem 1.2. Suppose that $5_1$ is obtained from $3_1^*4_1$ by a positive crossing change. Then by Theorem 1.5(4), we know that $s(3_1^*4_1) - s(5_1) \in \{0, 2\}$. On the other hand, $s(3_1^*4_1) = \sigma(3_1^*4_1) = 2$ and $s(5_1) = \sigma(5_1) = 4$ by Theorem 1.5(3). This is a contradiction. Therefore we have $d_G(3_1^*4_1, 5_1) \geq 2$. In fact, $d_G(3_1^*4_1, 5_1) \leq 2$ as shown in Figure 3, and hence $d_G(3_1^*4_1, 5_1) = 2$. We cannot use Theorem 1.4 to show this because $Q(3_1^*4_1, \delta) = Q(5_1, \delta) = \sqrt{5}$. We cannot also use Theorem 1.6 since $|s(3_1^*4_1, 5_1)| = |1| = 1$. Next we prove (2) by using Theorems 1.2 and 1.6 as follows. Since $V(3_1^*4_1; \omega) = -\sqrt{3}$ and $V(5_1; \omega) = -1$, if $d_G(3_1^*4_1, 5_1) = 1$, then $d_{+}(3_1^*4_1, 5_1) \equiv 0 \mod 2$ by Theorems 1.3, and hence $d_{-}(3_1^*4_1, 5_1) = 0$. However, as shown above, $d_{+}(3_1^*4_1, 5_1) = 1$ by Theorem 1.6. This is a contradiction.

**Example 3.2.**

1. $d_G(3_1^*3_1, 4_1) = d_G(3_1^*3_1, 4_1) = 3$.
2. $d_{+}(3_1^*3_1, 4_1) = d_{-}(3_1^*3_1, 4_1) = 2$. 

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(3) $d_+ (3_t \sharp 3_1, 4_1) = d_+ (3_t \sharp 3_1, 4_1) = 0$. By Theorem 1.6, $d_+ (3_t \sharp 3_1, 4_1) \geq s(3_t \sharp 3_1, 4_1) = -s (3_t \sharp 3_1) = 2$ since $s (3_t \sharp 3_1) = -4$ and $s (4_1) = 0$.

Suppose that $d_+ (3_t \sharp 3_1, 4_1) = 2$. By Theorem 1.3, $d_+ (3_t \sharp 3_1, 4_1) \equiv 1 \mod 2$ since $V (3_t \sharp 3_1; \omega) = (-1)^6 (\sqrt{3})^2$ and $V (4_1; \omega) = (-1)^1 (\sqrt{3})^6$. Then we have $d_+ (3_t \sharp 3_1, 4_1) = 1$. This is a contradiction. Thus we know that $d_+ (3_t \sharp 3_1, 4_1) \geq 3$. As shown in Figure 4, we know that $d_+ (3_t \sharp 3_1, 4_1) \leq 3$, $d_+ (3_t \sharp 3_1, 4_1) \leq 2$ and $d_+ (3_t \sharp 3_1, 4_1) = 0$, and hence $d_+ (3_t \sharp 3_1, 4_1) = 3$, $d_+ (3_t \sharp 3_1, 4_1) = 2$. Since $d_+ (K, K') = d_+ (K^*, (K')^*)$, $d_+ (K, K') = d_+ (K, K') = d_+ (K^*, (K')^*)$ for any two knots $K$ and $K'$, we have $d_+ (3_t \sharp 3_1, 4_1) = 3$, $d_+ (3_t \sharp 3_1, 4_1) = 2$ and $d_+ (3_t \sharp 3_1, 4_1) = 0$.

**Example 3.3.**

(1) $d_G (3_t \sharp 3_1, 7_3) = 3$.

(2) $d_+ (3_t \sharp 3_1, 7_3) = 0$.

We can obtain the above values by using the same argument as in Example 3.2. In fact, $d_+ (3_t \sharp 3_1, 7_3) \geq 2$ by Theorem 1.6. On the other hand, we have $d_+ (3_t \sharp 3_1, 7_3) = 1$ by Theorem 1.3 since $V (3_t \sharp 3_1; \omega) = (-1)^6 (\sqrt{3})^2$ and $V (7_3; \omega) = (-1)^6 (\sqrt{3})^6$ if we assume that $d_G (3_t \sharp 3_1, 7_3) = 2$. Thus $d_G (3_t \sharp 3_1, 7_3) \geq 3$ and we obtain $d_G (3_t \sharp 3_1, 7_3) \leq 3$ and $d_+ (3_t \sharp 3_1, 7_3) = 0$ as shown in Figure 5, and hence $d_G (3_t \sharp 3_1, 7_3) = 3$. However we cannot detect $d_+ (3_t \sharp 3_1, 7_3)$. (This value is larger than or equal to 2 by the above argument.)

**Remark.** Miyazawa has given a criterion of the Gordian distance of knots by using the HOMFLY polynomial in [8]. The value $d_G (3_t \sharp 4_1, 5_1)$ is also obtained by making use of the result. On the other hand, $d_G (3_t \sharp 3_1, 4_1)$, $d_G (3_t \sharp 4_1, 4_1)$ and $d_G (3_t \sharp 3_1, 7_3)$ cannot be obtained from it.

**Example 3.4.**

(1) $d_G (4_t \sharp 4_1, 3_1) = d_G (4_t \sharp 4_1, 3_1) = 3$.
We know that $d_G(4_1 \sharp 4_1, 3_1), d_G(4_1 \sharp 4_1, 3_1^*), d_G(4_1 \sharp 4_1, 5_2), d_G(4_1 \sharp 4_1, 5_2^*), d_G(4_1 \sharp 4_1, 6_3)$ and $d_G(4_1 \sharp 4_1, 6_3)$ are less than or equal.
to 3 since \( d_G(3_1, 0_1) = d_G(4_1, 0_1) = d_G(5_2, 0_1) = d_G(6_3, 0_1) = 1 \). Then we know that the equations (1), (2) and (3) are obtained from Theorem 1.4 since \( Q(3_1; \delta) = Q(5_2; \delta) = Q(6_3; \delta) = -1 \) and \( Q(4_1; \delta) = -\sqrt{5} \). These numbers are undecided in the table of I. Darcy [2]. We cannot detect them by using Theorem 1.2, 1.3 because of the values of the Jones polynomials and we cannot also use Theorems 1.6 to detect them since 

\[
\sigma(4_1^2 4_1) = 0, \ s(3_1) (or \ |s(3_1^*)|) = \sigma(3_1) (or \ |\sigma(3_1)|) = 2, \ s(5_2) (or \ |s(5_2^*)|) = \sigma(5_2) (or \ |\sigma(5_2^*)|) = 2 \]

and \( s(6_1) = \sigma(6_1) = 0 \).

**Example 3.5.** Let \( Y \) be a knot in the set \( \{ 3_1, 5_2, 6_2, 7_6, 8_7, 8_{11}, 8_{14} \} \) (see Figure 7). Let \( X = Y^* \). Then

1. \( d_G(X, 10^*_145) = d_{-+}(X, 10^*_145) = 3, \ d_{+-}(X, 10^*_145) = 0 \),
2. \( d_G(X, 10^*_154) = d_{-+}(X, 10^*_154) = 4, \ d_{+-}(X, 10^*_154) = 0 \),
3. \( d_G(X, 10^*_161) = d_{-+}(X, 10^*_161) = 4, \ d_{+-}(X, 10^*_161) = 0 \).

It is easily see that \( d_G(X, 0_1) = 1 \) and \( d_G(10^*_145, 0_1) \leq 3, \ d_G(10^*_154, 0_1) \leq 3 \) and \( d_G(10^*_161, 0_1) \leq 3 \). (If we change signed crossings in Figures 6 and 7, we obtain trivial knots.) Then we know that \( d_G(X, 10^*_145) \leq 3, \ d_G(X, 10^*_154) \leq 4 \) and \( d_G(X, 10^*_161) \leq 4 \). Now we show that \( d_{-+}(X, 10^*_145) \geq 3, \ d_{-+}(X, 10^*_154) \geq 4 \) and \( d_{-+}(X, 10^*_161) \geq 4 \). We can show that \( 10^*_145, 10^*_154 \) and \( 10^*_161 \) as in Figure 3 are quasipositive knots [11][12][14]. In fact, \( 10^*_145 \) and \( 10^*_161 \) are the closure of quasipositive braids 

\[
\sigma_2^3 \sigma_3 \sigma_1 \sigma_2 \sigma_3 \] and 

\[
(\sigma_1 \sigma_2 \sigma_3)^{-1} \sigma_2 \sigma_3 \sigma_1^{-1} \sigma_2 \sigma_3 \]

respectively and \( 10^*_154 \) is a positive knot with a positive diagram consists of 10 crossings and 5 Seifert circles as described in Figure 6.

![Figure 6](image)

Then by using a result of Shumakovitch [14], we know that \( s(10^*_145) = 4 \) and \( s(10^*_154) = s(10^*_161) = 6 \). On the other hand, \( s(X) = -s(Y) = -s(\sigma(X)) = -2 \) since each \( X \) is an alternating knot. (For each knot \( Y \), by a crossing change of the positive crossing signed in Figure 7, we have a trivial knot. Thus by Theorem 1.5(4), we know that \( s(Y) \geq 0 \).) Thus by using Theorem 1.6, we have 

\[
d_{-+}(X, 10^*_145) \geq \frac{s(X) - s(10^*_145)}{2} = 3, \ d_{-+}(X, 10^*_154) \geq \frac{s(X) - s(10^*_154)}{2} = 4 \]

and 

\[
d_{-+}(X, 10^*_161) \geq \frac{s(X) - s(10^*_161)}{2} = 4. \]

Thus we have 

\[
d_{-+}(X, 10^*_145) = d_G(X, 10^*_145) = 3, \ d_{-+}(X, 10^*_154) = d_G(X, 10^*_154) = 4 \]

and 

\[
d_{-+}(X, 10^*_161) = d_G(X, 10^*_161) = 4, \]

and hence 

\[
d_{-+}(X, 10^*_145) = d_{+-}(X, 10^*_154) = d_{+-}(X, 10^*_161) = 0. \]

However we cannot use Theorem 1.4 to detect them because 

\[
Q(10^*_145; \delta) = Q(10^*_154; \delta) = -1 \] and \( Q(10^*_161; \delta) = -\sqrt{5} \). (See Figure 8 for a value of \( Q(X; \delta) \).)

Here we give the following problem.

**Problem.** For any four knots \( K_0, K_1, K_2, K_3 \), does the next equality hold?
\[ \text{Figure 7} \]

\[ d_G(K_0 \sharp K_1, K_2 \sharp K_3) = \min\{d_G(K_0, K_2) + d_G(K_1, K_3), \; d_G(K_0, K_3) + d_G(K_1, K_2)\}. \]

In the first example, \( d_G(3_1, 0_1) = d_G(3_1, 5_1) = d_G(4_1, 0_1) = 1 \). Thus we have
\[ d_G(3_1, 4_1, 0_1) = \min\{d_G(3_1, 0_1) + d_G(4_1, 5_1), \; d_G(3_1, 5_1) + d_G(4_1, 0_1)\} = 2 \]
even if we cannot detect \( d_G(4_1, 5_1) \). (In fact, we know that \( d_G(4_1, 5_1) \geq 2 \) by making use of Theorem 3.1.)

We list signatures, special values of the Jones polynomial and the \( Q \) polynomial for knots with up to 8 crossings (Figure 8.) (Here we set \( a = \sqrt{-3} \) and \( b = \sqrt{5} \).)
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