ON PRIME KNOTS WITH SYMMETRIC UNION PRESENTATIONS

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Abstract. A symmetric union is a knot in the 3-space, obtained from a knot and its mirror image, which are symmetric with respect to a 2-plane in $\mathbb{R}^3$, by taking the connected sum of them and moreover by connecting them with some vertical twists along the plane. In this paper, we give a sufficient condition for a symmetric union to be prime.

Key words: Symmetric union; prime knot.

1. Introduction

A symmetric union was originally introduced by Kinoshita and Terasaka [2] and later, Lamm [3] generalized the definition. A symmetric union is known to be an example of a ribbon knot [1].

A knot $K$ is composite if there is a sphere $S$ intersecting $K$ transversally in two points, such that neither of the 3-balls bounded by $S$ intersects $K$ in a single unknotted spanning arc. The sphere $S$ is called a decomposing sphere for $L$. A non-trivial knot is prime if it is not composite. In this paper, we study a prime knot with a symmetric union presentation. In [6], we have given a sufficient condition for a symmetric union to be prime. We refine the result as follows.

Theorem 1.1. Let $K$ be a symmetric union with minimal twisting number one and $\~{D} \cup \~{D}^*(m)$, a symmetric union presentation of $K$. If $\~{D} \cup \~{D}^*(\infty)$ is a diagram of a trivial link, then $K$ is prime.

Throughout this paper, $\sharp\{X\}$ denotes the number of elements of $X$ for a finite set $X$. In Section 2, we shall give the definitions of a symmetric union and the minimal twisting number. In Section 3, we shall prove Theorem 1.1. In Section 4, we shall give some examples.

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2. The definition of a symmetric union

A symmetric union is defined as follows. (See [3] for the original definition.) Let $\mathbb{R}^3$ be the 3-space with $x$-, $y$-, and $z$-axes. Let $\mathbb{R}^3_+ = \{(x, y, z)|x > 0\}$ and $\mathbb{R}^3_- = \{(x, y, z)|x < 0\}$. Throughout this paper, a tangle denotes a disjoint union of two arcs properly embedded in a 3-ball. We denote the tangle made of $|m|$ half-twists along the $z$-axis as a diagram by an integer $m \in \mathbb{Z}$ and the horizontal trivial tangle by $\infty$ with respect to the $x$-axis as in Figure 1.

Definition 2.1. We take a knot $\tilde{K}$ in $\mathbb{R}^3_+$ and its mirror image $\tilde{K}^*$ in $\mathbb{R}^3_-$ such that $\tilde{K}$ and $\tilde{K}^*$ are symmetric with respect to the $yz$-plane $\mathbb{R}^2_{yz}$ as in Figure 2(a). Here we consider a diagram of a knot in the $xz$-plane $\mathbb{R}^2_{xz}$ and we denote the diagrams of $\tilde{K}$ and $\tilde{K}^*$ by $\~{D}$ and $\~{D}^*$, respectively. Each disk-arc pair of $T_0, T_1, \ldots, T_k$ as in Figure 2(a) denotes a diagram of the tangle 0. Then we replace the tangles $T_0, T_1, \ldots, T_k$ with tangles $\infty, m_1, m_2, \ldots, m_k$ as in Figure 2(b). (See Figure 3 for example.) Here we
assume that $m_i \neq \infty$ (1 ≤ i ≤ k). The resultant diagram is called a *symmetric union presentation* and we denote it by $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$.

If a knot $K$ has a diagram $\tilde{D} \cup \tilde{D}^*(m_1, \ldots, m_k)$, then the knot $K$ is called a *symmetric union*.

Here we define the minimal twisting number for a symmetric union which was originally introduced in [5] as follows.

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Definition 2.2. We call the number of non-zero elements in \( \{m_1, \ldots, m_k\} \) the twisting number for \( \hat{D} \cup \hat{D}^*(m_1, \ldots, m_k) \). The minimal twisting number of a symmetric union \( K \) is the smallest number of the twisting numbers of all symmetric union presentations to \( K \).

Remark 2.3. The minimal twisting number is an invariant of \( K \). The minimal twisting number of a two-bridge symmetric union is equal to either one or two [4]. We gave an example of a symmetric union with minimal twisting number two in [5].

3. Proof of Theorem 1.1

Proof of Theorem 1.1. In the case when \( m \) is an odd number, the primeness of \( K \) follows from the proof of Proposition 6.1 [6]. From now on, we use notations defined in [6]. We assume that \( K \) is composite in the case when \( m \) is an even number. Then it is enough to consider the case when \( \sharp\{d_t \cap K\} = 2 \) in the proof of Proposition 6.1 [6]. In that case, we have the following two cases:

(i) \( d_t \subset B \); or
(ii) \( d_t \subset \overline{B}^3 \).

In the case of (i), we have a connected summand of a trivial knot if \( \sharp\{d_t \cap K\} = 2 \) and \( m = 0 \) if \( d_t \cap K = \emptyset \). This is a contradiction. In the case of (ii), as in the proof of Theorem 1.2 [6], we take a simple arc \( \alpha \) on \( d_t \) which connects two intersection points of \( K \) and \( d_t \). Then we have a 2-component (split) link \( L \) by a surgery of \( K \) along \( \alpha \). Then it is easily seen that the linking number of \( L \) is non-zero since \( m \) is non-zero. This is a contradiction. Thus we know that \( K \) is a prime knot.

4. Symmetric unions of two-bridge knots

Example 4.1. We consider symmetric unions of two-bridge knots. We denote the following two-bridge knot by \( T(b_1, b_2, \ldots, b_{2d}) \) \( (d > 0) \). (Each \( b_i \) \( (1 \leq i \leq 2d) \) denotes the number of full-twists.)

We take a symmetric union of \( T(b_1, b_2, \ldots, b_{2d}) \) and its mirror image, as in Figure 5. We denote the resulting knot by \( B_m(b_1, b_2, \ldots, b_{2d}) \). In Figure 5, \( m \) represents vertically arranged \( |m| \) crossings, which are right-handed if \( m > 0 \) and left-handed if \( m < 0 \).

It is easily seen that \( B_{\infty}(b_1, b_2, \ldots, b_{2d}) \) is a trivial link. By using the formula of the Jones polynomial in [5], we know that the Jones polynomial of \( B_m(b_1, b_2, \ldots, b_{2d}) \) is as follows:

\[
V(t) = (-1)^m t^{-m} F(t) F(t^{-1}) + ((-1)^m - 1) t^{-m},
\]

where \( F(t) \) is the Jones polynomial of \( T(b_1, b_2, \ldots, b_{2d}) \). Since \( T(b_1, b_2, \ldots, b_{2d}) \) is the alternating knot, we know that the reduced degree of the Jones polynomial is \( \sum_{i=1}^{2d} d_i \) and then the minimal degree of \( F(t) F(t^{-1}) \) is \( -\sum_{i=1}^{2d} d_i \). If \( -m > \sum_{i=1}^{2d} d_i \geq 2 \), then the minimal
degree of $V(t)$ is $-\Sigma_{i=1}^{2d}d_i - m > 0$. In particular, $B_m(b_1, b_2, \ldots, b_{2d})$ is not amphicheiral and so the minimal twisting number of $B_m(b_1, b_2, \ldots, b_{2d})$ is non-zero. Thus we know that $B_m(b_1, b_2, \ldots, b_{2d})$ is a prime knot if $-m > \Sigma_{i=1}^{2d}d_i \geq 2$ by Theorem 1.1. We also find that the maximal degree of the Jones polynomial of $B_m(b_1, b_2, \ldots, b_{2d})$ is $\Sigma_{i=1}^{2d}d_i - m$ if $-m > \Sigma_{i=1}^{2d}d_i \geq 2$. Then by Corollary 7.2 [6], we know that the minimal twisting number of $B_{m_1}(b_1, b_2, \ldots, b_{2d}) \sharp B_{m_2}(b_1, b_2, \ldots, b_{2d})$ is two if $-m_1 > \Sigma_{i=1}^{2d}d_i \geq 2$ and $-m_2 > \Sigma_{i=1}^{2d}d_i \geq 2$, where $K_1 \sharp K_2$ denotes the connected sum of two knots $K_1$ and $K_2$.

References


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